Origins of Contemporary Philosophy:

Studies on Philosophy of Mathematics

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To Ingolf Max.

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Vera philosophiae methodus

nulla alia nisi scientiae naturalis est.

Franz Brentano

SUMÁRIO

Presentation

Contemporary philosophy has been characterized by the schism between “analytic” and “continental” philosophers. However, recent decades have shown signs of rapprochement between these traditions. This situation has motivated a growing interest in the study of the philosophy of the second half of the nineteenth century and the beginning of the twentieth. The result of these efforts has been the delimitation of a sphere of research that, under the title “Origins of Contemporary Philosophy,” is beginning to restudy an important chapter in the history of recent philosophy left ostracized as a result of the rupture mentioned above.

Identifying the birth of contemporary philosophy involves critically reviewing ideas that were widely disseminated at the beginning of analytic philosophy and phenomenology. In both cases, rigorous historiography has been replaced by “creation myths” that, although relevant to determining the identity that each of the currents attributes to itself, ignore a web of complex systematic and historical relationships.

Today, it is no longer possible to ignore the variety of themes and problems involved in the gestation of the two currents and still present, in a linear narrative, a complex, multidimensional history that is not yet sufficiently known. Despite the fact that, since the 1950s, various authors have sought to recover the affinities between Frege's motivations and those of Husserl, there is still a need to understand how these motivations led to the construction of what is contemporary philosophy. The revision of such simplistic approaches requires a rereading of the whole of nineteenth century philosophy, proposing a history of philosophy that is properly historiographical and properly philosophical, as opposed to a romantically idealized vision.

This is the scope of the book *Origins of Contemporary Philosophy: Studies on the Philosophy of Mathematics*, published here by "Apolodoro Virtual Edições" as a special issue of the series "Rationality, Intentionality and Semantics," which brings together a presentation of the research group "Origins of Contemporary Philosophy" (GPOFC/PUC-SP) by Prof. Dr. Evandro O. Brito (UNICENTRO) and three papers resulting from research carried out by the German researcher Prof. Dr. Julia Franke-Reddig (University of Geneva/University of Siegen) and the Brazilian researchers Prof. M.e. Ernesto Maria Giusti (UNICENTRO) and Prof. Dr. Arthur Heller Britto (Pontifical Catholic University of São Paulo).

Origins of Contemporary Philosophy: A Presentation of the GPOFC Research Area

1. Introduction

A *sui generis* phenomenon specific to contemporary philosophy is the radical schism between two traditions, the so-called "analytical" philosophy and the so-called "hermeneutic-phenomenological" philosophy. The characteristic note of this phenomenon is that, for the first time in the history of philosophy, two research traditions have emerged that do not simply represent opposing positions on certain philosophically relevant topics but do not talk to each other at all. In this sense, contemporary philosophy is marked by a drastic rupture in the philosophical community that inhibits communication among its members. The last few decades, however, have shown signs of rapprochement between these traditions. This situation has motivated a growing interest in studying the philosophy of the second half of the nineteenth century and the beginning of the twentieth. The result of these efforts has been the delimitation of a sphere of research that, under the title "Origins of Contemporary Philosophy," is beginning to reexamine an important chapter in the history of recent philosophy left ostracized as a result of the aforementioned schism. As Porta argues, identifying the birth of contemporary philosophy involves critically reviewing ideas that were widely disseminated at the beginning of analytical philosophy and phenomenology (Porta & Brito, 2023, p. III).

In both cases, rigorous historiography has been replaced by "creation myths" that, although relevant to determining the identity that each of the currents attributes to itself, ignore a web of complex systematic and historical relationships.

2. The Myths

Analytic philosophy is said to have emerged in England as a reaction to the British version of Hegelian idealism led by Russell and Moore, who incorporated the logical advances made by Frege into their thinking. Similarly, phenomenology was an extemporaneous creation by Husserl, motivated by Brentano's reflections on intentionality and Frege's criticism of the psychologism of his first work on the foundations of arithmetic.

Both myths mask the true motives and authors involved in a process that was not the work of isolated individuals but of countless thinkers from the beginning of the nineteenth century. Today, we can no longer ignore the variety of themes and problems involved in the gestation of the two currents and present, in a linear narrative, a complex, multidimensional history that is not yet sufficiently known. Although since the 1950s various authors have sought to recover the affinities between Frege's motivations and those of Husserl, the work of understanding how these motivations led to the construction of what is contemporary philosophy is still a necessity.

The revision of such simplistic approaches requires a rereading of the whole of nineteenth-century philosophy, proposing, in contrast to a romantic view, a history of philosophy that is properly historiographical and properly philosophical.

This romantic view interprets nineteenth-century philosophy as a heroic overcoming of the impasses of the Enlightenment, after Kant, through a few isolated names such as Nietzsche, Marx and Freud. However, it is mistaken as history, as philosophy and as the history of philosophy, because it ignores the philosophical production of the period. The fundamental problems remain unanswered in this model: When did contemporary philosophy emerge and what characterizes it? Or again, when and why did modern philosophy "end"?

3. The Aims of the GPOFC

The study developed by the Group aims to reanalyze the philosophy of the period. This reanalysis brings changes to the philosophical scene, with the incorporation of numerous authors who have not yet been adequately studied, either as thinkers with original ideas or as essential actors in a philosophical transition that took place involving elements of this tradition, as well as certain ruptures within it. Authors such as Stuart Mill, Bolzano, Herbart, Trendelenburg, Lotze, Stout, Brentano, Marty, Stumpf, Meinong, Fischer, Dilthey, Schleiermacher, Cohen, Natorp, Windelband, Rickert, Nelson and Rehmke have yet to be properly incorporated into philosophical historiography, which narrates the historical development of contemporary philosophy, and related to each other. The aforementioned reanalysis also encourages a review of the periodization and classification of the various authors and philosophical currents.

This rereading implies a change of focus and problems in understanding this period of philosophy, which involves identifying and reinterpreting themes of epistemology, logic, philosophy of language and metaphysics, which allow us to understand how plural contemporary philosophy is. The thematic form of the research appears to be dispersed due to the numerous authors. However, the issues they raise are recurrent – the realm of nonreal objectivity, anti-psychologism, representations without objects and terms without reference, among others.

The starting point and presupposition is Kantian transcendental philosophy and its derivations. Analytic philosophy and hermeneutic phenomenology arise from the same movement of ideals, the roots of which lie in post-Hegel German-language philosophy. This movement must have its own profile outlined, in addition to the already established and widely studied anti-rationalist, positivist and Hegelian reactions.

The aim of the group is therefore to study and establish the relationships between analytic philosophy and phenomenological-hermeneutic philosophy in the process of their formation from the perspective of establishing their common roots, as well as the moment of their reciprocal isolation. The working hypothesis is that analytic philosophy and phenomenological hermeneutics represent, from the point of view of the history of philosophy, and despite all their undeniable differences, a common "turn" that can be characterized as the shift from the concept of validity (*Geltung*) to the concept of meaning or significance (*Sinn*, *Bedeutung*). The thesis of the existence of a systematic turn of the same nature implies, and is based on, the thesis, purely historical-philosophical, of a common origin whose process refers to a fourth post-Hegel line of development already mentioned. It is on the basis of this twofold observation that we must understand the process of isolation of traditions in the twentieth century, as well as their eventual fates, on which the fate of philosophy as a whole ultimately depends.

The concentration of nineteenth-century philosophy as a whole on the theme of meaning is a phenomenon that cannot be properly understood if we stay within the field of philosophy. Both the division between analytics and continentalists and the establishment of their common roots through attention to the establishment of meaning and signification as a specific sphere are essentially linked to the so-called "identity crisis" of philosophy, and this, in turn, is linked to the situation of science in the nineteenth century. Attention to this phenomenon will allow us to understand why a study of the origins of contemporary philosophy can only be developed as a properly interdisciplinary investigation. This point deserves more detailed consideration.

Since, in principle, the history of philosophy as a whole is closely linked to the state of science, the important thing here is to establish what are the peculiar characteristics of this relationship in the nineteenth century. Now, if the eighteenth century represents the constitution of the project of mathematical understanding of the universe, which has as its consolidated result the establishment of the *Naturwissenschaften*, the nineteenth century brings with it, in addition to the extension of the *Naturwissenschaften* from physics to chemistry, biology and physiology, three phenomena that transcend their scope:

a. on the one hand, the emancipation of particular disciplines and their consolidation as autonomous sciences (history, psychology and, as we shall see, linked to it, linguistics, sociology, pedagogy, etc.).
b. on the other hand, and on the basis of the above, the extension of the attempt at scientific understanding from the sphere of nature to the sphere of “spirit” (Geist). The nineteenth century was the century of the *Geisteswissenschaften* (literally: sciences of the spirit). The idea of “*Geisteswissenschaften*” is closely linked to that of cultural sciences (*Kulturwissenschaften*), moral sciences (moral sciences) and/or social sciences (social sciences) (the most common expressions in the Anglo-Saxon sphere), and human sciences (*sciences humaines*) (the most common expression in the French-speaking sphere), although each of these terms expresses a specific nuance and is linked to a particular tradition.

c. the emergence of new sciences, different in their objectives from the established sciences, which will urgently raise the question of epistemological and methodological monism-pluralism in science. Should all science follow the model of the *Naturwissenschaften* or are there possible sciences that respond to another model?

The fact that science as a way of knowing extends its claim to knowledge to the totality of reality, thus including areas that had traditionally been considered objects of philosophy (such as "spirit" and the soul), has urgently raised the question of the extent to which philosophy should not simply disappear or be replaced by science. This question is what is traditionally referred to as an "identity crisis." If we take into account the ultimate root of this crisis – namely, I repeat, the pretension of science to account for the totality of reality, and the fact that, since the "Critique of Pure Reason" in the nineteenth century, it is no longer possible to defend the autonomy of philosophy from science by attributing to the latter the study of a supernatural reality – it is clear that the only way out of this crisis is to discover another sphere of nonreal objectivity, the sphere of the objectivity of meaning, significance and value.

4. The *Psychologismusstreit*

What we have briefly and very generally explained can be studied in detail if we focus on psychology, which plays an essential role in this process. Until the eighteenth century, psychology was part of philosophy. During the nineteenth century, however, psychology became independent of metaphysics and raised its claim to be an autonomous science. This, however, did not come without a troubled history. If psychology is fighting for its autonomy from metaphysics, it will also have to fight for its autonomy from the subsequent emergence of physiology, and even for its inclusion or not in the framework of the *Geisteswissenschaften*, and eventually, if this question is answered positively, for the clarification of its role. Should psychology be a natural science, albeit of a specific type, or should it be a *Geisteswissenschaft*? And if it is a *Geisteswissenschaft*, should it properly be the basis or foundation of all the others?

Given that, obviously, an affirmative answer to this question threatens the autonomy and peculiar specificity of the various disciplines that make up the *Geisteswissenschaften*, the question concerning the relationship between psychology and the *Geisteswissenschaften* will open one of the decisive chapters of a much larger process called the "quarrel over psychologism" (*Psychologismusstreit*), in which, ultimately, what was at stake was the affirmation or denial of the thesis of identities between reality and objectivity. If senses, meanings and values are not physical objects, then they must be "psychic objects" and therefore psychology must deal with them. On the other hand, if senses, meanings and values are something objective but not real, neither physical nor psychic, then they require a *sui generis* type of scientific approach.

If the problem of psychologism arose with regard to the *Geisteswissenschaften* and their relationship with psychology, it did not focus on them, but had as its essential moment the relationship of psychology with logic, semantics and, through these disciplines, linguistics, sociology and pedagogy.

a. Nineteenth-century mathematics produced the so-called "arithmetization of calculus," which led to a gradual abandonment of the Kantian thesis that space-time intuition was the foundation of this discipline. This abandonment of the foundational role of intuition in mathematics led to what is known as "logicism," i.e., the program of grounding mathematics in logic. This gave rise to the idea of continuity between logic and mathematics and, consequently, the notion of "formal sciences" in the contemporary sense. This grounding, in turn, required a clarification of the very concept of logic that also eliminated any reference to intuition from this discipline. The result of this process was the need to separate logic from psychology.

b. The separation of logic and psychology ends up leading, by an internal dynamic, to a separation of logic and semantics. If truth and its formal relations are not psychological processes or events, then neither can what is true, or, in other words, the so-called "truth bearer," be. The meaning of a statement cannot therefore be considered a psychological entity. If, until now, language studies had focused on the discussion of its origin and acquisition, and therefore had a properly psycholinguistic nature (to put it in contemporary terms), the delimitation of a properly semantic problematic allowed for a new (nonpsychological) perspective on language, which promptly linked the already existing syntactic consideration of grammar to give rise to linguistics.
c. Something similar happened with sociological phenomena. While society was initially considered to be a mere collection of individuals and therefore subject to study by psychology, little by little the awareness has grown that there are psychological phenomena that are not individual but social and, even more so, that there are properly sociological phenomena that cannot be reduced to social psychology or even less to individual psychology. Finally, something similar must be said about pedagogy.

Combining the various elements described so far, we can summarize our results in three closely interrelated points:

a. The *Psychologismusstreit* was not only decisive for *Geisteswissesnchaften*, logic, linguistics and sociology, but also for philosophy as a whole.
b. The discussion in philosophy will be essentially linked to its discussion in other sciences and spheres of knowledge.
c. The end of the *Psychologismusstreit* in philosophy will bring an end to the identity crisis that will allow philosophy to maintain its place in the sphere of knowledge and culture in general, but now clearly delimiting it from another mode of knowledge, namely the scientific.

Looking back, we can say that the identity crisis of philosophy opens up four alternatives:

a. Philosophy disappears as a legitimate activity in the sphere of knowledge, as an archaeological relic of the history of human culture (positivism).
b. It reformulates its problems in the context of natural science; in other words, it becomes a "naturalized philosophy."
c. It dissolves into psychology, which, depending on how this psychology is conceived, would imply that it is an aspect of the naturalization program or an element of the discussion around *Geisteswissenschaften*.
d. It finally establishes as its object a sphere of nonreal objectivity: meaning, significance and value.
e. From Kant onwards, it becomes clear that the survival of philosophy necessarily implies detaching it from the study of reality, be it sensible (due to the success of science) or suprasensible (due to the failure of metaphysics), in order to focus on questions of validity (*Geltung*), epistemology, ethics and aesthetics. In this sense, it could be said that it has been clear since Kant that the survival of philosophy requires the establishment of the

idea of nonreal objectivity. However, as the nineteenth-century process shows, this gain is not assured until the difference between real and unreal objectivity as such is established, because the study of validity as a second-order study is always threatened with falling into a first-order study. The harsh polemic against psychologism will be decisive in this sense to avoid these dangers and thus definitively consolidate the Kantian deception. However, precisely because of this, it necessarily forces us to go beyond Kant and rethink the central place of the theory of knowledge in philosophy, replacing it with philosophical semantics. If the clear delimitation of a sphere of nonreal objectivity in relation to real objectivity makes it possible, on the one hand, to distinguish questions of validity from questions of fact, on the other hand, it also opens up the possibility (more precisely) of posing a question of nonreal objectivity as more primitive than the question of the objectivity of validity, namely the question of the objectivity of meaning. If the question of the validity of scientific statements is a question of nonreal objectivity, in contrast to real objectivity, then the question of the validity of scientific statements must be preceded by the question of the meaning of these scientific statements. Establishing a plane of nonreal objectivity, therefore, while allowing the problem of validity to be placed in its specificity, will force the reformulation of this problem by making the question of the objectivity of meaning and significance precede it.

5. Conclusion

This is, therefore, the historical-philosophical and scientific background that guides the Research Group on the Origins of Contemporary Philosophy (GPOFC), as well as the debate here on the philosophy of mathematics in this context.

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Facets of Ernst Mach's Influence on Moritz Schlick[1]

1. Historical and Philosophical Background

Friedrich Albert Moritz Schlick is widely recognized today as the founder and "integrational figure" (Ingolf Max) of the Vienna Circle. However, before embarking on his philosophical career, he studied physics and distinguished himself as the favorite student of his supervisor, Max Planck, under whom he earned his doctorate for a thesis on the *Reflections of Light in an Inhomogeneous Layer* in 1904 (MSGA I.2). During this period, he also engaged with Planck's critical assessment of Mach's philosophical perspectives (see e.g. Stadler 2021). This seems fitting, as Schlick himself noted years later that his study of physics was driven by philosophical interest (MSGA I.2: pp. 11f.). Consequently, it is unsurprising that Schlick delved deeper into psychological and philosophical studies over the following years (MSGA I.1: pp. 9f.), culminating in his habilitation at the University of Rostock in 1911 with a thesis on the *Nature of Truth according to Modern Logic*[2].

In Rostock, he remained until 1921, when he was invited to the University of Kiel. After a brief year in Kiel, he received an offer from the University of Vienna to

[1] This chapter was written by Profa. Dr. Julia Franke-Reddig (University of Geneva/University of Siegen).

[2] Unfortunately, the new edition of this book, which was part of the MSGA I.4, has not been published until today. Important preparatory works for it were recently published within the MSGA II.1.1.

assume the chair previously held by Ernst Mach (Stadler 2021: p. 199; Stadler 2015: p.301; MSGA I.6: pp. 11ff.). Despite later receiving an invitation from the University of Bonn, he chose to stay in Vienna until the tragic end of his life in 1936. In Vienna, he initiated the meetings of the so-called 'Schlick-Zirkel' and became the chairman of the Ernst-Mach-Association, against whose resolution due to political struggles in Austria during the 1930s he fought until the end (Stadler 2015: pp. 615ff.).

What was Schlick's philosophical standpoint? Current literature suggests that the 'early' Schlick espoused critical realism during his years in Rostock. However, he soon shifted this position after his arrival in Vienna, influenced by Ludwig Wittgenstein, and maintained a more positivist stance in his later works (see e.g. Neuber 2018: Chapter 1, Section 2). In a self-description published posthumously in 1953, Schlick wrote that he sought to justify the construction of a consistent and purely empirical approach (*Selbstdarstellung* 1931). Building on this, his philosophy has recently been labeled as "Consequent Empiricism" (Friedl 2013, especially pp. 19f.) and a method of "consequent philosophizing" (Max 2023 and 2022). Schlick continuous:

> Previous forms of empiricism, from Sextus Empiricus to Mill and Mach, were neither pure nor consequent because they could not provide a satisfactory account of logic and mathematics—the 'rational.' However, Schlick's new empiricism precisely starts from the understanding of mathematical thinking and its application to reality.[3]

[3] „Frühere Formen des Empirismus, von Sextus Empirikus bis zu Mill und Mach, waren weder rein noch konsequent, weil sie von Logik und Mathematik — dem „Rationalen" — keine

Considering this background, it is evident that Schlick saw himself as following a tradition in which Ernst Mach played a significant role. Moreover, the examination of Mach's work permeated his entire career. How should we assess the influence of Machian Philosophy on Schlick's work? On the surface, one might assume that the ‘early’ Schlick rejected Mach’s standpoint, while the ‘late’ Schlick supported it. However, Friedrich Stadler (2021) points out that Schlick’s early critique of Mach obviously arose under the influence of Planck and concludes that it was “just one more sign that within Logical Empiricism there was a group of philosophers which preferred a more realistic position, later called critical or structural realism” (Stadler 2021: p. 200). However, Stadler’s analysis of Schlick’s philosophical relations to Mach only considers his first major work published in 1918 and later writings. In the following, I am going to supplement this argumentation regarding Schlick’s earlier writings, especially those recently published from his estate, as well as his philosophical interpretation of the special theory of relativity from 1915. This will provide us with a broader view of the development of Schlick’s position on Mach’s philosophy. In doing so, I will introduce the thesis that the ‘early’ Schlick not only denied Mach but was also influenced by him. This entails consideration of three different facets of this influence: first, the way Schlick formulated his own philosophical standpoint in demarcation from Mach (Section ii); second, how his examination of Machian thoughts was an important background for his acquaintance with Einstein (section iii);

befriedigende Rechenschaft geben konnten. Der neue Empirismus Sch.s geht aber gerade von dem Verständnis des mathematischen Denkens und seiner Anwendung auf die Wirklichkeit aus.“ (Selbstdarstellung 1931) and finally, his incorporation of a Machian concept into his own philosophy (section iv).

In my dissertation titled *On the Continuity and Independence of Moritz Schlick's Philosophy of Science*, I argue that the common differentiation between an 'early' and a 'late Schlick' is incorrect and that the philosophy of the 'early' Schlick already leaned towards a positivist position (Franke-Reddig 2024). This paper serves as a supplementary thesis, highlighting the ways in which Schlick developed central aspects of his late philosophy much earlier during his early phase in Rostock. Moreover, I aim to show that further investigations into the impact of Mach's positivist position, especially on Schlick's early writings, could indicate that his consideration of the Machian standpoint persisted throughout his entire philosophical career, subsequently providing a deeper understanding of Schlick's philosophical development.

2. Schlick's Rejection of Mach's Positivism

The *General Theory of Knowledge* ("*Allgemeine Erkenntnislehre*", hereinafter referred to as GTR) can be regarded as Schlick's early main work. It was written between 1911 and 1915, and first published in 1918 (MSGA I.1: p. 1). Consequently, it can be attributed to the "early" Schlick. In his paper from 2021, Stadler clearly outlines Schlick's evolution of thought regarding Mach's positivism, from the standpoint initially presented in this work to his subsequent perspective in Vienna:

> […] Moritz Schlick (1882-1936), started from a critical-realist position in his *Allgemeine Erkenntnislehre* (*General Theory of Knowledge*, 1918/1925) before, influenced by Wittgenstein, he preferred a more 'positivist'

> version of his philosophy (with growing criticism from Max Planck and Einstein). In this respect, the early Schlick was closer to Boltzmann than to Mach […]. (Stadler, 2021: p. 197)

What Stadler means is Schlick's clear demarcation from Mach within the GTR, particularly within his "Critiques of the Immanence Thoughts" (MSGA I.1: pp. 496-542). In this context, Schlick perceives Mach (alongside Avenarius) as the exponent of what he terms the "philosophy of immanence" (MSGA I.1: pp. 492ff.). The essence of this philosophy, in Schlick's words, is defined by the "dogma of the identity of the real with the given" (MSGA I.1: p. 540). That means that he characterizes Mach's strict positivism as the dogmatic stance asserting that reality can only be ascribed to the given, i.e., what is actually perceived by the senses[4].

He is employing this (fictitious) Machian position to outline his own philosophical standpoint as a middle ground between the philosophy of immanence on one side and Kantian transcendental philosophy as its counterpart on the other side. This intermediary approach involves rejecting the Machian "dogma" while simultaneously negating the unknowability of things in themselves. How does this function? Schlick dismisses the Kantian concept of 'things in themselves' and redefines it by asserting his own perspective: he designates those objects as real but not given and unequivocally identifies them as knowable (MSGA I.1: pp. 480-496). This serves as a prime example of how a philosopher might be influenced by philosophical ideas without entirely subscribing to them; the impact lies in the self-demarcation from Schlick to Mach and Kant. Schlick

[4] Whether this is an adequate reading of Mach, cannot be discussed within this context. At this point for us is not relevant, what Mach truly meant, but how Schlick interpret him.

defines things-in-themselves as realities not given to oppose Mach, while asserting their knowability as a rejoinder to Kant, ultimately defining his own conception. By rejecting Mach's viewpoint, which, as Schlick asserts, only considers the actually given as real, and the Kantian thesis that things in themselves are unknowable, he acknowledges both philosophers' perspectives. The influence of Mach and Kant on Schlick lies in this act of delineation.

Moreover, within the GTR, Schlick's investigation of the Machian standpoint, as well as his examination of Kantian philosophy, is noticeably more extensive compared to other philosophers he was commenting on. One reason for this prominence compared to other philosophical views might lie in their connection to the sciences. Schlick noted much earlier, in 1910/11, that the most significant aspect of Kantian Epistemology was the examination of exact scientific thinking through philosophy, stating, "the same applies, no matter how different the task solved is, of Mach's epistemology, etc." (MSGA II.1.1: p. 363f.). In a later writing from 1915, we can read that "beneath the Kantian," no other contemporary philosophical movement ("*philosophische Richtung*") is connected as closely to exact sciences as positivism; subsequently, he discusses Mach (Schlick 1915: p. 43). Considering this judgment on positivism and Kantianism, Schlick's aim to develop a philosophy that truly does justice to scientific practice might be seen as a reason why he gave Mach and Kant, in particular, so much space in his GTR. Taking this into account, one might assume that Mach could be seen as one of the most significant influences on Schlick, especially in his early career. Furthermore, one could argue that Schlick's scientific career began when he received Ernst Mach's "Mechanic" as a gift for his Matura in 1900 (MSGA II.1.2: p. 371; Mach 1883). Thus, this was one of the first scientific and philosophical studies Schlick ever encountered. Additionally, he read works by Mach, years before the completion of the

GTR, as we will see, especially regarding his demarcation from Mach's philosophy. There, for example, we find one of his sample lectures from 1909/10 titled "Idealism and its Refutation," which he prepared for his habilitation. Here, he comments on Mach, stating that the claim that only sensations exist has a dogmatic character (MSGA II.1.1, p. 290). He further adds that Mach is unable to avoid the presumption of the transcendental: the principle, which only allows us to consider the directly given and does not permit any transcendental assumption, would lead to solipsism (MSGA II.1.1, p. 290). Schlick concludes:

> In his [Mach's] efforts to save himself from it, he then falls into metaphysical idealism and is unable to remain true to his own critical intentions and premises.[5]

In his later writings, as in the GTR, Schlick employs the term 'transcendent' (*transzendent*) or 'transcending' (*transzendieren*) to signify 'assuming something beyond the given' or 'presuming entities not given as real' (see e.g. MSGA I.1: pp. 445, 483f., 487f., 489, 492, 497 and 543). Taking this into account, the previously mentioned passage contains preparatory considerations for Schlick's conception of things-in-themselves within the GTR that I mentioned earlier: he argues that to avoid solipsism or metaphysical idealism, we must attribute reality to objects that were not perceived. This constitutes an important aspect of his concept of things-in-themselves, and as we have

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[5] „Bei den Bemühungen, sich von ihm zu retten, gerät er dann in den metaphysischen Idealismus und ist nicht imstande, den eigenen kritischen Absichten und Voraussetzungen treu zu bleiben." (MSGA II.1.1: pp. 290f.)

observed, he formulates it in his examination of Mach, even at a very early stage of his philosophical career.

So, the conclusion here is that the ‘early’ Schlick worked his philosophical way through Mach, with demarcation from him being a central point in his philosophical development. Nevertheless, Stadler is correct in noting that Schlick underwent a transition to an anti-metaphysical 'positivism' and emphasizes his “ever stronger commitment to Mach” in his later period in Vienna (Stadler 2021: p. 198). However, I would like to add at this point that this is especially true for his explicit commitment to Machian positivism. As I pointed out in my dissertation, the position of the ‘early’ Schlick was not as anti-positivist as commonly assumed: Although he distanced himself from Mach’s philosophy, his critical-realist standpoint was not strictly realism; rather, it implicitly pointed towards a positivist direction (Franke-Reddig 2024).

However, there are even more facets of Schlick’s early philosophy in which the examination of Mach has been very important for him and might have exerted significant influence. In the following section, I am going to argue for the importance of Schlick’s early encounter with Mach for his acquaintance with Einstein and his theory.

3. The Machian Background of Schlick’s Interpretation of the Theories of Relativity

Schlick’s article on the philosophical relevance of the principle of relativity (“*Die philosophische Bedeutung des Relativitätsprinzips*”, hereinafter referred to as RP) has been recently republished along with several other writings under the title “*Schriften zur Relativitätstheorie*” (Engler 2020). It is one of the earliest published philosophical examinations of the groundbreaking works of Albert Einstein. Under the title “On the Electrodynamics of Moving

Bodies", Einstein published his theory of relativity in 1905("*Zur Elektrodynamik bewegter Körper*", Einstein 1905). The general theory of relativity was first published in 1916 (Einstein 1916). Schlick's RP is from 1915; therefore, he refers only to what we now call the *special* theory of relativity, not its generalization. Within this article, we find lengthy passages in which Schlick examines Mach, especially a formulation of the principle of relativity that, as Schlick states, follows from Machian positivism. He reformulates this principle as follows:

> The fundamental idea of positivism, to explain only the perceived as real, to construct the world solely from immediately given "elements," has often led to the assertion: since only relative motions are perceptible, they alone are real; absolute motions do not exist and therefore cannot have any physical significance, no physical effect.[6]

Later in the text, he formulates this principle not only as a philosophical assumption but also as the physical demand for the equality of all reference systems (Schlick 1915: p. 51). The relevant question for Schlick at this point is whether this postulate is equivalent to the principle of relativity contained in Einstein's special theory of relativity. His answer to this question is succinctly formulated: "No!" (Schlick 1915: p. 44). As a reason for this, he

[6] „Der Grundgedanke des Positivismus, nur das Wahrgenommene für wirklich zu erklären, die Welt allein aus unmittelbar gegebenen »Elementen« aufzubauen, hat oft zu der Behauptung geführt: da nur relative Bewegungen wahrnehmbar sind, so sind auch nur sie allein wirklich, absolute Bewegungen existieren gar nicht und können daher auch keine physikalische Bedeutung, keine physikalische Wirkung haben.“ (Schlick 1915: p. 44)

declares that positivism must assert relativity for all movements, while Einstein’s principle of relativity only applies to uniform translational movements (Schlick 1915: p. 44). Within this context, he mentions that Einstein has attempted to generalize his theory so that the principle of relativity is not only valid for inertial systems but also for accelerated movements (Schlick 1915: p. 48). He states that this generalization of the theory would lead to an “extraordinary fundamental simplification of the worldview”. Subsequently, he cites Einstein, who claims epistemological reasons for his view (Schlick 1915: p. 48; Einstein 1915).

Thereupon, with a reference to Petzoldt, Schlick discusses the question of whether a generalization of Einstein’s theory would represent a “great triumph of Mach's philosophy” (Schlick 1915: p. 50). One could argue for this, as the general principle of relativity might be viewed as the empirical confirmation of the principle of relativity derived from Mach’s works. However, Schlick consistently denies this for three reasons, one of which is particularly interesting in our discussion. Schlick claims:

> It has indeed turned out that even Einstein's extended theory fails to fully implement the idea of limitless relativity of accelerations; not every arbitrary frame of reference is equally valid according to it, as the Machian principle unequivocally demands.[7]

[7] „es hat sich nämlich herausgestellt, daß auch Einsteins erweiterte Theorie den Gedanken der *schrankenlosen* Relativität der Beschleunigungen nicht durchzuführen vermag; nicht *jedes* beliebige Bezugssystem ist nach ihr gleichberechtigt, wie das Machsche Prinzip es unbedingt fordern muß.“ (Schlick 1915, p. 51)

Continuing, he adds that Einstein himself has demonstrated that a solution to this problem for arbitrarily moving coordinate systems cannot even exist; consequently, it is impossible to interpret Einstein's theory as a validation of Mach's postulate of relativity (Schlick 1915: p. 51).

Schlick sent an exemplar of RP to Einstein at the beginning of December 1915, which posed a challenge for Einstein, as Engler recently pointed out (Engler 2020: p. XV). For Einstein, Machian ideas could be seen as 'heuristic guiding' (see e.g. Hentschel 1987: p. 30), and he later introduced the concept of a "Machian Principle" himself, as implemented in his generalized theory (Hentschel 1987: p. 47, see also Engler & Renn 2018: p. 143). By this point in 1915, he had already "achieved his goal of deriving generally covariant field equations in the meantime," (Engler 2020: p. XVf.) which seemed, on the one hand, to contradict his "Machian Heuristic" and, on the other hand, to refute Schlick's argumentation (Engler & Renn 2018: p. 140). Upon receiving Schlick's RP, he promptly sent a favorable response to Schlick:

> From a philosophical perspective, there seems to be nothing nearly as clear written on the subject. Moreover, you possess a complete mastery of the material. I have no objections to your explanations.[8]

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[8] „Hochgeehrter Herr Kollege!
Ich habe gestern Ihre Abhandlung erhalten und bereits vollkommen durchstudiert. Sie gehört zu dem Besten, was bisher über Relativität geschrieben worden ist. Von philosophischer Seite scheint uberhaupt nichts annähernd so Klares über den Gegenstand geschrieben zu sein. Dabei beherrschen Sie den Gegenstand materiell vollkommen. Auszusetzen habe ich an Ihren

Einstein's mail also contained the news regarding the generalized theory of relativity:

> Your comments on the general theory of relativity are entirely correct, as far as this theory has been correct so far. The new discovery is the result that there exists a theory compatible with all previous experiences, whose equations are covariant under arbitrary transformations of space-time variables.[9]

Subsequently, Einstein invited Schlick for a visit to discuss his new theory from both a physical and, importantly, a philosophical standpoint. According to Engler, it strongly suggests that this meeting took place by the end of December (Engler 2020: pp. XIIIf.). In his introduction to Schlick's writings on the theory of relativity, Engler posits the thesis that during this meeting, Einstein and Schlick discussed the so-called "Hole Argument," which, from a Machian perspective, posed a philosophical problem for Einstein regarding the ontology of his theory (Engler & Renn 2018: p. 135ff.). Subsequently, Einstein replaced the "Hole Argument" with Schlick's so-called "Point Coincidence Argument" (Engler 2020: p. XVIf.; Engler & Renn 2018: pp. 140ff.). This argument was implied in Schlick's

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Darlegungen nichts." (Engler & Iven & Renn 2022: Einstein an Schlick, 14. Dezember 1915)

[9] „Auch Ihre Bemerkungen über die allgemeine Relativitätstheorie sind ganz richtig, soweit diese Theorie bisher überhaupt richtig war. Das neu Gefundene ist das Resultat, dass es eine mit allen bisherigen Erfahrungen vereinbare Theorie gibt, deren Gleichungen beliebigen Transformationen der Raum-Zeitvariablen gegenüber kovariant sind." (Engler & Iven & Renn 2022: Einstein an Schlick, 14. Dezember 1915)

Principle of Coincidences within his General Theory of Knowledge, which he had recently completed. This principle was of great importance to Einstein, as Engler states:

> The adoption of Schlick's principle of coincidences could finally complement the important role that the Machian principle played ontologically for the general theory of relativity, by giving epistemological precedence to material events over the structure of spacetime.[10]

Shortly after his first meeting with Einstein, on the recommendation of Erich Becher, Schlick began writing a philosophical essay on the special and general theory of relativity, which was well-received. He expanded its content to publish his book on *Space and Time in Contemporary Physics.* Within just four years, the book had been republished in four editions Physics ("*Raum und Zeit in der gegenwärtigen Physik*", MSGA I.2). Einstein and Schlick maintained close personal contact, and their correspondence continued until 1933. Throughout those years, Einstein remained an important scientific and philosophical contact for Schlick.

What does all this have to do with the influence of Mach on Schlick? It is undisputed that Einstein's theories were a crucial test for Schlick's philosophy of science and that Einstein himself played a significant role in supporting Schlick's philosophical career. I hope to have clarified, through the story of the beginning of Schlick's and

[10] „Die Übernahme des Schlick'schen Prinzips der Koinzidenzen konnte schließlich auch die wichtige Rolle, die das Mach'sche Prinzip ontologisch für die allgemeine Relativitätstheorie spielte, ergänzen, indem es materiellen Ereignissen nun auch erkenntnistheoretisch den Vorrang vor der Struktur der Raumzeit gab." (Engler 2020: pp. VIIf.)

Einstein’s acquaintance, how essential the philosophical examination of Mach was in this regard: Schlick, who was very familiar with Mach’s ideas, recognized their importance for the heuristic of Einstein’s theories. He closely discussed Machian positivism as well as his critiques of Newtonian physics from philosophical perspective, highlighting that Einstein would not have been able to establish his theories without having examined positivist philosophy beforehand (Schlick 1915: p. 44). Einstein confirmed and praised this observation (Engler & Iven & Renn 2022: Einstein an Schlick, 14. Dezember 1915)[11]. Considering that Schlick had already completed his work on the GTR, in which he presented his delineation from Mach, we could speculate that Schlick’s stronger commitment to Mach may have begun already at this time, prior to 1922, as suggested by Stadler, who claims it was “inspired by Wittgenstein and Carnap”. (Stadler, p. 198). This challenges the strict distinction between a critical realist ‘early Schlick’ in Rostock and a positivist ‘late Schlick’ in Vienna. One indication of this might be that Schlick's critique of Mach in *Space and Time in Contemporary Physics* is much less severe than in RP (see MSGA I.2: “*Raum und Zeit in der gegenwärtigen Physik*” especially chapter X: “*Beziehungen zur Philosophie*” pp. 267ff.). However, this is not the place to exhaustively discuss this hypothesis, as it would require extensive analysis, especially of Schlick’s recently published early writings from his estate, as well as his forthcoming correspondence, therefore, this remains a task for future research.

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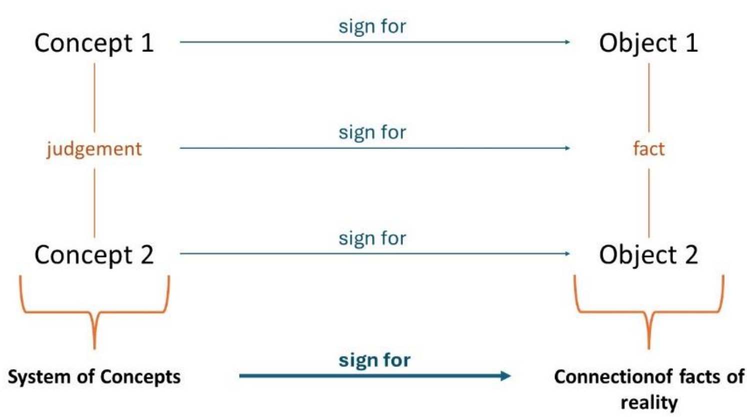
[11] He writes: „Auch darin haben Sie richtig gesehen, dass diese Denkrichtung von grossem Einfluss auf meine Bestrebungen gewesen ist, und zwar E[rnst] Mach und noch viel mehr Hume, dessen Traktat über den Verstand ich kurz vor Auffindung der Relativitätstheorie mit Eifer und Bewunderung studierte.“

4. The Principle of Reduction

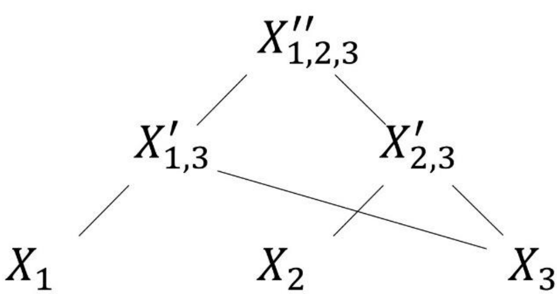
This final section serves to present a third facet of Mach's influence on Schlick, as well as to reveal one aspect of the relationship between Schlick's philosophy of science and mathematical thinking. Therefore, it is important to consider Schlick's 'systems of concepts' that he develops within the GTR. In the following, I will provide a brief explanation of these systems. For a more detailed examination, see (Franke-Reddig 2023, section 2) and (Franke-Reddig 2024).

Schlick notes that a scientific theory is a connection ("*Zusammenhang*") of knowledge (in the sense of "*Erkenntnisse*", MSGA I.1: p. 195). This connection is to be expressed by a system built up by concepts, so he speaks of "Systems of Concepts", which are connected to each other by judgments ("*Urteile*"). Each individual concept within such a complex conceptual system is to be understood as a sign for an object of reality, while the objects of reality are, in turn, connected to one another by facts. The judgments of the system represent signs for those facts. In whole, the systems of concepts are thereby signs of the complex factual connections of reality (MSGA I.1: §§ 3, 5, 8, 9, 11).



How is such a system of concepts constructed? At the base, we can find the "concretely defined concepts" ("*konkret definierte Begriffe*"), which are not to be further analyzed (MSGA I.1: pp. 201f.), they are the "ultimate characteristics" ("*letzte Merkmale*"), which can only be defined by intuitively showing their meaning and in no other way (MSGA I.1: pp. 199f.).

On the upper levels of such a hierarchically constructed systems of concepts, there are only implicit defined concepts, which means that they are defined by the axioms of the system (MSGA I.1: p. 208). Schlick is inspired by David Hilbert's axiomatization of geometry from 1899 in his use of implicit definition (Hilbert 1899; MSGA I.1: §7). These implicitly defined concepts encompass all concepts at the lower levels, which means that the concepts from the lower steps of the system can be derived from the higher-level concepts. In the following illustration, for example, the concept $X'_{1,3}$ subsumes the concepts $X_1$and X_3, and conversely, the concepts $X_1$and X_3 can be deduced from $X'_{1,3}$. The concept $X''_{1,2,3}$ represents the axiom of the system: by using the principles of logic and mathematics, it must be possible to constitute the entire system of concepts solely from it.



In such a system, from bottom to top, we progress from special to general concepts. We do this by utilizing our capacity for knowledge (see e.g. MSGA I.1, pp. 139, 162f., 164, 166). These systems of concepts are to be understood as a mathematical reconstruction of scientific knowledge: they aim to elucidate the structure of scientific theories. Within this context, it is interesting to recognize the similarities between Schlick's systems of concepts and Hilbert's method of axiomatic thinking (Hilbert 1917). In this sense, Schlick's philosophy of science can be seen as a conception of knowledge that "provides a satisfactory account of logic and mathematics," as cited above: he adopted, for example, this purely mathematical concept to construct a logical system aimed at revealing the nature of human knowledge. Although he did not use the term 'implicit definition' after 1926, I was able to show in my dissertation that these systems of concepts and the underlying idea of implicit definition were not only ideas of the 'early' Schlick that he rejected in later years: it was furthermore a preliminary work for his later elaborated thesis, that only structures are knowable and communicable, what was called the "structural thesis" by Johannes Friedl (Friedl 2013: Chapter 4, Section 2). So, these considerations have been central in Schlick's early as well as in his later works, and within this context, we can find at least hints of Machian philosophy: a central aspect of these systems of concepts lies in the rule that as we move from downward to upward within such systems, the number of concepts has to be reduced for each step. Stadler (2021) already pointed out that with assumptions like that, Schlick already formulated the principle of economy according to the characterization in the 1929 manifesto in *Space and Time in contemporary physics* (Stadler 2021, p. 197). But, as we will see now, we can also find this within the GTR: In questioning the nature of knowledge, Schlick defines a principle of reduction, which we can find in several passages of the GTR. He writes, for example,

> the way described, the number of appearances explained by the same principle is continually increasing, and consequently, the number of principles necessary to explain the totality of appearances is decreasing.[12]

Unsurprisingly, Schlick denies Mach's principle of economy in the GTR as it was – in his words – not the right way to express the essence of science (MSGA I.1: p. 318). Nevertheless, at this point, he admits:

> There is a correct core underlying it, and the reader of the preceding chapters cannot doubt what he is looking for: Knowledge consists of designating things in the world completely and unambiguously through a minimum of concepts; economizing with the smallest possible number of basic concepts - therein lies the economy of science.[13]

With this, he directly connects the Machian concept with a central aspect of his philosophy of science. And this passage was already contained within the 1918 edition of the GTR, hence within the writings of the so-called early

[12] „auf die geschilderte Weise wird die Zahl der Erscheinungen, die durch ein und dasselbe Prinzip erklärt werden, immer größer, und demnach die Zahl der zur Erklärung der Gesamtheit der Erscheinungen nötigen Prinzipien immer kleiner." (MSGA I.1: pp. 162f.)

[13] „Ihm liegt ein richtiger Kern zugrunde, und dem Leser der vorhergehenden Kapitel kann es nicht zweifelhaft sein, worin er zu suchen ist: Das Erkennen besteht ja darin, die Dinge der Welt durch ein Minimum von Begriffen vollständig und eindeutig zu bezeichnen; mit einer möglichst geringen Anzahl von Grundbegriffen auszukommen – *darin* besteht die Ökonomie der Wissenschaft." (MSGA I.1: p. 318)

Schlick. But what we can see here is not just demarcation; it is alignment. Furthermore, we can already find this alignment in Schlick's first lecture at the University of Rostock in 1911/12. There, he states that it was the objective of knowledge to designate as many concepts as possible, eventually all with a minimum of signs (MSGA II.1.1: p. 436). Subsequently, he comments as follows:

> This truth, that the aim of all knowledge is to represent a maximum of facts with the smallest possible number of concepts, we call the Principle of the Economy of Thought. So it essentially suggests that the drive for knowledge is most fully satisfied when thinking is as economical as possible in the use of concepts.[14]

Subsequently, Schlick states, with reference to Mach and Avenarius, that the described principle plays an essential role for positivism, but that the positivists have failed to formulate it sharply enough and have extended it too broadly (MSGA II.1.1: p. 437). Of course, there lies a disagreement with Mach, but despite that, we can notice at least an appreciation for positivistic thoughts and the implementation of a core principle into Schlick's own philosophical conviction. This principle is, as also stated by Stadler (2021), central for the philosophy of the Vienna Circle and, as mentioned above, for Schlick's later constructed

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[14] „Diese Wahrheit, dass das Ziel aller Erkenntnis darin besteht, ein Maximum von Tatsachen durch die kleinstmögliche Zahl von Begriffen darzustellen, nennen wir das Princip der Oekonomie des Denkens. Es sagt also gewissermassen aus, dass der Erkenntnistrieb am vollkommensten sich befriedigt fühlt, wenn das Denken in der Verwendung der Begriffe so sparsam ist wie möglich." (MSGA II.1.1: pp. 436f.)

‘structural thesis’. He did not adopt this during his time in Vienna; he already formulated it, as shown here, in his earliest writings. At this point, a differentiation between Schlick’s ‘early’ and ‘late’ philosophy would not be justified. For this reason, it is clear that the rise of Schlick’s version of logical empiricism goes further back than his acquaintance with Wittgenstein’s philosophy in Vienna.

At the end, we do not know if Schlick was inspired by Mach in the development of his principle of reduction. It is reasonable to assume that he may have been influenced by many other philosophers in this regard. However, he attributes this idea, which played a central role in his early and late philosophy, to Machian (and Avenarius') philosophy at this early point in his career. Considering this, we can conclude that in ‘early Schlick's’ philosophy, there lies not only rejection of Mach but also at least cautious approval.

5. Conclusion

In the preceding discussion, I have argued that the impact of Mach's ideas on Schlick's earliest works is much greater than initially apparent. In this context, I have examined three different aspects of Schlick's philosophical engagement with Mach's work. Firstly, I explained how Schlick formulated his central concept of knowable things-in-themselves in contrast to Mach, not only in his GTR but much earlier than previously thought. In the following section, considering the historical context, I proposed the hypothesis that Schlick's reevaluation of Machian positivism may have begun earlier than recently assumed. Finally, I demonstrated that the relationship between the ‘early Schlick’ and Mach was not solely one of rejection but also included agreement with what he termed the “true core” of Mach's principle of economy. These three aspects offer compelling reasons to abandon the strict differentiation between an ‘early’ and a ‘late’ Schlick, but could not be thoroughly and conclusively explored here. The reasons for this are twofold: firstly, such investigations would require much more space, and secondly, important writings of Schlick are still awaiting publication by the Moritz Schlick Research Centre in Rostock, including forthcoming editions of Schlick's correspondence. Nonetheless, I hope to have revealed here that further investigations, particularly into Schlick's early writings, hold the promise of providing new insights into the philosophical development of the founder of the Vienna Circle, and consequently, into the evolution of analytical philosophy in general.

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Geometry and the concept of space in Brentano[15]

1. Introduction

The aim of this article is to explore some texts that point to a Brentanian philosophy of geometry and space in Brentano's late, though not final, writings. The main source available to develop such a study can be found in Brentano's correspondence with the Italian mathematician, philosopher and polymath Giovanni Vailati, dating from March 4, 1900 to March 1902, and eventually in his writings on space from 1906-08. As we will show, Brentano is still firmly embedded in what we can call the modern Euclidean tradition, and thus refuses to recognize any mathematical - or philosophical - value to non-Euclidean geometries. This means that, unlike the formalist orientation that would be established under Hilbert's influence, he sees the theory of geometric knowledge as dependent on and linked to a theory of space and our knowledge of it. In other words, as we will argue in this paper, there is a point in common between the Brentanian approach and that of his neo-Kantian contemporaries, namely the commitment of both to the thesis that the intuition of space - although differently conceived - plays a fundamental role in the constitution of mathematical knowledge.

Rather than a systematic study of these themes, our intention is simply to point out some of the aspects in which the reflection on mathematics and geometry appear in Brentano, and justify the study of this topic. In fact, after going through the usual bibliography on the author, it seems to the attentive reader that, unlike his neo-Kantian

[15] This chapter was written by Prof. Ernesto Maria Giusti (UNICENTRO/Brazil).

contemporaries, and the physiologists and empiricists who developed extensive reflections on the nature of mathematical knowledge, Brentano would not have conceived a theory on the subject. The literature on the Austrian philosopher is particularly lacking in studies on these themes and in research into the mathematical background of Brentano's thinking.

Looking at the historical period, however, we must be persuaded that the situation probably does not correctly describe the reality of Brentano's interests. Husserl, for example, in the Philosophy of Arithmetic expressly declares his debt to Brentano's observations on mathematics. If we consider the later development of the so-called “Brentano School”, we will also see how the interest in the philosophy of mathematics stems directly from insights and theses developed from Brentano's teachings, whether in lectures or texts. Carl Stumpf, for example, one of his first disciples, illustrates this situation.

What we have just said above is, on the one hand, a justification for our investigation, but it is also a glimpse of the consequences we are seeking to find: to point out, and contribute to satisfying, a gap that is still present in studies on Brentano. However, a broader contribution of these theses concerns how to understand the historical period that resulted in contemporary philosophy and which still requires an effort of understanding that has not been made in its entirety.

2. Mathematics in Brentano's studies

An initial approach to the proposed topic should first convince the reader of the need to read and understand Brentano's notes on mathematics. Such a note may seem idle, but it stems from the current state of Brentanian research, and how the specialized results of this research are appropriated by the mainstream of contemporary philosophical reflection. In line with recent developments in the philosophy of mind and the renewal of phenomenology, numerous other aspects of Brentano's philosophy are more studied and produce a greater volume of literature. Certainly his relationship with the phenomenological project, and his concept of intentionality are the most investigated, providing contributions to philosophy of mind, theory of knowledge, mereology, etc. to this day. Other parts, such as his moral theory and his theory of the perception of time, as well as the transition to so-called reism in his final period, have received more attention. However, what Brentano had to say about mathematics, and especially geometry, and even the relationship between the concept of space and geometry, is much less studied.

There are reasons for this, and they are not the effect of any bias on the part of his readers, but reflect a concrete state of affairs, since these problems receive a briefer and more tentative treatment than other issues in the Brentano corpus. In fact, none of Brentano's major works are dedicated exclusively or primarily to these themes, and they appear only timidly in the tables of contents of the published volumes and in the themes of his numerous lectures and lessons. However, these texts do exist and deserve to be studied, mainly because they deal with a series of central issues of various philosophical movements of the second half of the 19th century and the beginning of the 20th, contemporary to Brentano, but also to Frege and Husserl.

In fact, reflection on the philosophical foundations of mathematics, especially with regard to the construction of alternatives to the Kantian appeal to intuition, decisively marked the emergence of contemporary philosophy, motivating everything from the algebra of logic in its Germanic and British traditions to Bolzano, Frege, Peano and Hilbert. The critical reformulation of the Kantian theory of the synthetic a priori and its replacement by the search for a constitutive but non-transcendental objectivity is decisive not only for the emergence of the so-called analytical tradition, but also its so-called continental or, more appropriately, phenomenological-hermeneutic counterpart. Concurrent with the philosophical impulses of this transformation, there are also those that are properly mathematical, where the various revolutions broaden the methods and objects of mathematics. In the 19th century, we should consider at least two successive revolutions in geometry, that of projective geometry and that of non-Euclidean geometries, as well as topology and modern logic at the end of the period. Establishing the consequences of these revolutions and thinking philosophically about the new mathematics is an enterprise that unites thinkers as varied and fundamental as Frege, Dedekind, Russell, Helmholtz, Mach, Boltzmann, Poincaré, Riemann, Husserl, Stumpf, Cohen, Natorp and Cassirer, to name but a few.

There is a long line that runs through the 19th century and then the beginning of the 20th, in the foundations of mathematics, defined by the reaction to Kant, either to rework Kant in a positive way or, more often, to criticize Kant and offer alternatives. It is in this process that the philosophy of mathematics is constituted as a specialized and defined branch in the field of philosophical investigation. Although the presence of mathematics, as an ideal and as a model, has been decisive throughout the previous history of Western philosophy, it is only at this point that these authors delimit a set of questions which, taken together, constitute a new research project, and form this new field, the philosophy of mathematics and, in particular, as far as we are concerned, the philosophy of geometry. A representative, but not exhaustive, list of this constellation of questions includes: 1) is geometry an a priori science? Or is it just an empirical, physical science, albeit a very generalized one, an a posteriori science, based on mere generalizations of experience (Mill) or of our best scientific practices, especially in their application to physics? 2) If it is a priori, is it a priori in Kant's sense, based on synthetic judgments built on the foundations of intuition? 3) or is it a priori in an analytical sense, and should we identify mathematics ultimately with logic? The answer to this set of problems itself determines others, such as the more general and metaphysical question about the nature of space and its relationship to geometry. This is a question about the very object and method of geometry, that is, its nature. Is it a science of space, and if so, what kind of space (physical, perceptual space, or an absolute space analogous to Newton's)? Or is it just a science about abstract spatial structures (Riemann) or an "empty" science about mere logical relations (Hilbert)?

3. Brentano and geometry - a Kantian problem

The Kantian content, both in the form and in the content of the questions presented above, which persists to this day and makes them still relevant, is put in a classic way by Coffa (1994, p. 14) in his

To the Vienna Station - The semantic tradition from Kant to Carnap:

> Faced with the Scylla of asserting that 2 + 2 = 4 is empirical and the Charybdis of explaining it through the operations of pure intuition, semanticists chose to turn the boat around and try to find a better route. That there is a priori knowledge - even of the synthetic type - was indubitable to all of them; but most semanticists regarded the appeal to pure intuition as a hindrance to the development of science. (...) it came to be recognized that pure intuition must be excluded from the a priori sciences and that

> consequently the Kantian picture of mathematics and geometry must be replaced by some other. (1994, p. 14)

Brentano is aware of many of these problems and the solutions put forward, and to be convinced of this you only have to browse through his works, including the major ones such as Psychology from an Empirical Point of View, where he explicitly discusses, for example, Locke, Berkeley, Mill, Kant, Bolzano, Poincaré, Helmholtz etc., as well as various mathematicians of the time. Brentano's style, which always leads him to make exhaustive analyses of the alternatives offered by his predecessors, has the quality of readily revealing his sources.

However, Brentano's positive pronouncements on mathematics and geometry in this work remain laconic and indirect. At this point, the help of the discussion of some themes explicitly connected to the foundations of geometry in the correspondence with Vailatti sheds decisive light on these gaps. However, in order to understand the answers given by our author to these questions, it is necessary to list the elements that are intertwined in them. These different elements must be distinguished before they can be synthesized and related to his philosophy (or the different periods of it) in a more complete way. We must therefore first identify a problem concerning the relationship of space to geometry, and this depends on how Brentano understands the concept of space himself. Then there is a problem concerning the nature of the axioms of geometry, and their epistemological status. Finally, a last problem concerns the specific use of the concept of limit or boundary in the descriptive analysis of space (and its relation to Aristotle). For the purposes of this exposition, we will leave aside the discussion of the last point, and concentrate on the first two.

And here we can nuance the dilemma presented by Coffa, since it is not true that the abandonment of the

Kantian appeal to intuition was universally accepted in the period in which Brentano was writing. Frege, and various neo-Kantians such as Cohen and Natorp, are notable exceptions. Even if they undergo radical changes, the concept of intuition still plays a central role in explaining the nature of mathematical knowledge. We believe, however, that Brentano can also be understood in a certain way as someone who attributes a positive role to a certain use of intuition, although perhaps not in a strictly Kantian sense, as we will see below.

If we recall the geometrical revolutions mentioned above and examine their reception, we see that non-Euclidean geometries were not seen as a fatal blow to the philosophy of mathematics of Kantian origin, but rather, in many cases, stimulated its renewal. Many philosophers rejected the universal validity or truth of non-Euclidean geometries on the basis of Kantian-type arguments, appealing either to the Euclidean nature of the space of our experience, like Frege, which implies the truth of Euclidean geometry “for us” and therefore its truth tout court; or, like the neo-Kantians, renewing the concept of intuition in the light of the new mathematical and scientific developments of the century. Even a staunch anti-Kantian like Brentano seems to adopt a similar strategy to explain geometrical knowledge. It is in this sense that we should understand Chisholm and Corrado's statement when they say that: “(...) we may say that according to Brentano our knowledge of pure mathematics and of pure geometry begins with experience but does not arise out of experience.” (CHISHOLM, 1982 p. 4).

The statement points not to Brentano's subscription to an empiricist or empirical theory of space, as opposed to a nativist or priorist theory of space, options drawn by the terms of the discussion posed by Brentano's contemporaries. Rather, it points precisely to the limitation of understanding the problem in terms of this opposition, and the need to overcome this duality for a correct explanation of our geometrical knowledge. A passage from a 1906 booklet makes this position crystal clear:

> But without wanting to choose here between nativism and empiricism, I rather observe only that from what has been said it is surely possible to speak, at least in a broader sense, of a "space of sensation". However, we speak about spaces (spaces of time) also referring to the time continuum and in the newest geometry we find the name "space" applied to fictions of any great number of dimensions. (BRENTANO, 1988, p. 70)

Another way of formulating the same thesis consists of the apparently paradoxical statement that space is "neither inside us nor outside us". It is this reformulation of the problem of space by Brentano that relates it to a central motif of Brentano's entire philosophy, namely the problem of "our relationship with the world", which Brentano seeks to answer with his central and distinctive concept of intentionality.

Thus, the Brentanian turn consists of exchanging the question "what is space?", the answer to which was demanded by the Kantian-inspired philosophies of geometry, for the question "what is our relationship to space?". Although mediated by intentionality, this question, by emphasizing the role of the subject or consciousness in grounding space, contains inescapably Kantian accents, and Brentano does not abandon them in his answers. Although it is an exaggeration to call Brentano a Kantian, it should be clear that it is equally inappropriate to characterize Brentano as purely rationalist or purely empiricist - or any other label.

The 1906 text (1988, pp. 99-107) in which he discusses the "nativist, empiricist and anoethical" theories of "our representation of space" illustrates this Brentanian tendency, starting with the reference to our "representation of space" as the locus where we should investigate the foundations of geometry. The quoted text directly confronts the question, formulated first in terms of an investigation into the origin of our representation of space (analogous to the procedure he would adopt, in ethics, in his investigation "into the origin of our concepts of right and wrong") in which Brentano asks "where does our representation of space come from?"

The answer he offers is simple: "it comes from the common man". Now, this statement does not refer to any naive trust in the delivery of phenomenal data, but should rather be understood as referring to a conception of the functioning of "common sense" in terms that are both Lockean and Aristotelian. It is these that justify the statement that "our knowledge of space begins with experience", but is not exhausted by it. Brentano says

> Even Locke (...)shows himself to be at one with Aristotle in his results in this regard. He counts spatial extension among those simple ideas which, in contrast to the sensory qualities, are imparted to us through several of the senses. He is unmistakably thinking here not of a pure sensation of space but of sensations of concreta in which we find spatial as well as qualitative determinations." (p. 100)

Next, however, he adds that for "the most radical empiricists", by whom he means both Berkeley and Mill, a separation has to be made between our perceptual experience, understood as the first and raw data of our experience, and the resulting "experience of space" itself. Brentano is interested in pointing this out because it implies that, in empiricism, there is no representation of space in us that precedes things, and what we have is a perception of things themselves in their spatial relations, and never “of” space. Empiricism fails to justify the origin of our representation of space, in other words, because it lacks precisely this representation of space as its own concept or original representation.

Brentan's argument for refining this criticism combines the classical argument for the relativity of sensation with an appeal to the discussion on metric relations, which came into vogue especially after Gauss's geodesic discoveries. He starts from the trivial observation that the metric relations present in sensory experience do not correspond to those of the external world. The moon appears to be a few palms or a tower centimeters when seen from a distance by the human eye. The construction of adequate measurements is the result of a publicly reviewable process that cannot be entrusted to mere subjective perception, but presupposes the cooperation of the community of subjects in the form of scientific investigation: ““There is no fixed, transferable standard of measure. Thus, the length of a section of skin is not the length that is presented by the sensation of touch” (1988, p. 100). The preoccupation with measurement here reveals a turn typical of the spirit of the time, which also informs Frege's reflections on geometry, and which derives not from non-Euclidean geometries, but from the second geometric revolution we have mentioned, that of projective geometry and later topology. Not coincidentally, these are the areas of mathematics that explore the invariant spatial relationships between objects, leaving aside the metric ones. For these two authors, we believe that it is projective geometry that makes any form of radical empiricism impossible in the foundation of geometric knowledge. Although this is not the time to explore this topic, this is due to the fact that projective geometry enjoys, for both Brentano and Frege, greater respectability than non-Euclidean geometries, which are mere fictions in the best case, or simply false in the worst. Projective geometry, on the other hand, by exploring concepts such as the principle of duality, maintains its consistency with the space of Euclidean geometry, while at the same time exposing the difficulty of resorting to a merely sensible conception of intuition - and even an a priori intuition of Kant's type. The representation of the so-called “points at infinity”, the subject of Frege's doctoral thesis in the same year as the publication of Psychology from an Empirical Point of View, is an example of this role.

So Brentano is fully aware of his position in this discussion, and how it goes directly back to Kant and more specifically to the “Transcendental Aesthetics” of the Critique of Pure Reason. And, to the same extent, he firmly rejects that it implies a return to the Kantian alternative:

> In Germany (a) position has grown (...) out of a very legitimate opposition to Kant's doctrine of an a priori pure intuition of space that is supposed, as subjective form, to become a receptacle for all phenomena of outer sense. If we imagine all such phenomena removed, then, according to this doctrine, space- infinite in length, breadth and depth-would still remain as something absolutely incapable of being thought away.” (1988, p 101)

Here, we find a clear reference to the arguments of the section entitled “Metaphysical Exposition of the Concept of Space” of the “Transcendental Aesthetics” of Kant's first critique. The subsequent comment illustrates Brentano's familiarity with and use of mathematics to criticize Kant:

> Kant's psychological observations here were of no value, and he was also in the wrong when he made the possibility of geometrical reasoning

> dependent on the limitation of reasoning about plane continua of three dimensions. Mathematics since Riemann has ignored this limitation and has shown that topoids of arbitrary numbers of dimensions can be subjected to mathematical treatment. (1988, p. 101)

Reading these passages should be enough to establish the terms in which Brentano understands the problem. In the subsequent discussion of this work, Brentano reviews the solutions proposed by various authors, and why they fail in one way or another. Among these, we find Helmholtz, Boltzmann, Mach and Lotze (Brentano considers the latter "particular types of empiricists").

> In the controversy between the so-called nativists and the so-called empiricists concerning the origin of our presentation of space it is therefore nativism that is without doubt to be given preference. Not however nativism in certain unscientific guises, where insufficient account is taken of the contribution of association, habit and experience (...)". (1988, p. 103)

Thus, having exhausted the usual possibilities, Brentano seeks a "third way" or, to paraphrase Trendelenburg, a "neglected alternative" that allows us to explain the nature of space by examining the origin of our representation of space, which provides a foundation for the explanation of geometric knowledge in such a way as to escape both naive empiricism and the various forms of Kantianizing mathematical apriorism. However, it is not a question of simply abandoning apriorism or rejecting the need for our geometric knowledge, but of reformulating the concept of space in such a way as to take into account the contributions that the sciences, both natural, especially physiology and psychology, and mathematical, provide for an adequate understanding of it. Here, we believe, lies a point of unequivocal contact between the Brentanian strategy and the reformulation of the concept of intuition promoted by the neo-Kantians, for example by Hermann Cohen in his essay on the infinitesimal principle (COHEN, 1999) or by Natorp (NATORP, 1910), and whose proper study has yet to be done.

4. Geometry and the Correspondence with Vailati

It is curious that, in the examination of Brentano's reception, statements like the one in the passage just quoted have led Brentano to be accused of “positivism”. Oskar Kraus' posthumous defense of this accusation attests to the fact that this is not a secondary aspect of Brentano's school, but a central one. And it's curious because at another point, it was Brentano who declared that “I would like to distance Vailati from his positivism” (Letter to Anton Marty, May 15, 1900).

That both Vailati's and Brentano's thought have been called “positivist” is testimony to the fact that there are similarities between the two, as well as differences. These are revealed in the discussion of particular themes that soon lead to larger points of disagreement. The starting point for the epistolary exchange was a discussion about an article by Thomas Heath on the theory of parallels or directions according to Aristotle (HEATH, 1899). The writing of the famous historian of ancient mathematics and editor of Euclid leads Brentano to ask to examine the nature of mathematical proof. His first question is whether Euclidean (type) proofs, i.e. by the auxiliary use of diagrams, are demonstrations as such. For Brentano, the discussion about the possibility of a non-circular proof of the fifth postulate, the so-called postulate of parallels, requires an answer to this problem.

However, from his very first letter, Brentano presents an unusually Kantian defense that the existence of parallel lines is something we grasp based on our intuition of the nature of space. The most delicate point of this thesis, and one of the elements that is properly Brentanian, is that what we intuit is not space directly, but its nature.

Thus, although it uses the vocabulary of intuition, this is not a “direct and immediate contact with a particular” in the Kantian sense, but the elaboration of the nature of space (for example by accepting the postulate that determines the existence of parallel lines) from what is given and where, as we saw earlier, the “from” is more important than the “given” of experience.

It is therefore in keeping with the letter of the text and the spirit of the problem to understand “intuition” not in purely Kantian terms, but rather in those of a “source of knowledge” (in a sense close to the neo-Kantians already mentioned, such as Cohen and Natorp, but also to that of Frege (2009)) not as a direct reference to sensible experience or a set of data from it.

But let's return to the starting point of the correspondence, Thomas Heath's article on the theory of parallels or directions according to Aristotle. In the discussion of an enigmatic passage from the Posterior Analytics, I, 74a12, Aristotle uses the term “grafein” (γράφειν) . There are at least two possible interpretations of this term: to draw or to prove. It is ironic that the idea of “mathematical construction”, used in the title by Heath, resolves the disagreement because Kantian construction is precisely a type of proof by diagrams. Brentano, like Frege and most of his contemporaries, commits a skewed reading of Kantian philosophy of mathematics, since he almost exclusively privileges Transcendental Aesthetics to the detriment, for example, of the Transcendental Doctrine of Method. It is noteworthy that almost all the discussion of the role of intuition in Kant's foundation of geometry ignores the central role he gives to construction, and ostensive construction, as opposed to the symbolic construction of arithmetic - “geometrical knowledge results from the ostensive construction of concepts in intuition”. Thus, there is a whole Kantian context to the discussion that is ignored, as usual. Although usual, such a strategy implies trying to solve by the concept of intuition functions that are discussed by Kant in terms of “geometric construction”. But, returning to the text, there are, according to Brentano, three possible interpretations for the discussion of parallels in this passage, which correspond to three independent but interconnected questions: 1) Is a proof of the existence of parallel lines necessary? 2) Is it possible to prove the existence of parallel lines? 3) Is it possible to draw parallel lines? This is not exactly the typical discussion of the 19th century, which asked more directly whether it is possible to prove the Fifth Postulate (called by Brentano the Eleventh Axiom in the correspondence) and whether it is true. If the Fifth Postulate is true, then, for Brentano, Euclidean geometry is also true. This presupposes answering, in turn, what the nature of the Fifth Postulate is (and, consequently, of the axioms in general) and that's where the discussion ends.

In correspondence, these two types of questions, historical and systematic, are confused. We will conclude this article with a presentation of some of its main themes, in order to illustrate the previous statements in the discussion of a concrete mathematical example by Brentano and Vailati.

5. Brentano's answers

The correspondence between the two philosophers begins, with Heath's article as the occasion, with a discussion of a classic problem in the history of geometry, the status of Euclid's 5th postulate, that is, whether it can be proved, or whether its truth must be admitted as fundamental, which, of course, also involves the very truth of the postulate of parallels. Here, this problem is mediated by Aristotle, and the three authors, Heath, Brentano and Vailati, agree in interpreting the Stagirite as saying that "Aristotle did not believe that a true, logically valid proof of the existence of parallels was possible" (CHISHOLM, 1982, p. 7). Both Brentano and Vailati interpret this, however, not as determining adherence to the other alternative in the dilemma, but as requiring an adequate theory of axioms and their role in mathematical knowledge.

Thus, the disagreements between the authors concern systematic questions about the nature of axioms, and it is these that drive the epistolary. We will try to understand them by following the thread of the text. As is usual in the progressive analysis to which Brentano submits concepts, his initial starting point is the "common understanding" of a concept, or how the common man understands the concept of "parallels".

The type of answer I receive when I query laymen in geometry is always this: Parallels are straight lines which "run along side of each other [neben einander laufen]", or "straight lines which have the same direction without being parts of one and the same line" (...) [or] "straight lines which are everywhere equidistant from each other" (1982, p. 7).

The appeal to the concept of "direction" as a more basic concept from which it would be possible to analyze that of "parallels" was usual in the mathematics of the second half of the century: it is the same example used by Frege in §64 of the Fundamentals of Arithmetic. However, adopting the definition of parallel lines by the concept "direction" and attempting a proof inevitably leads to circularity, since parallels are defined as "lines that have the same direction" and, at the same time, it is assumed that if two lines have the same direction, then they are parallel. There is a clear circularity in the reasoning, which not only fails to resolve the problem, but also obscures the real problem. For the fact that two lines that have the same direction are parallel "is not self-evident, and requires proof. And proof requires the assumption that parallel lines exist." (1982, p. 16). In terms of the Kantian vocabulary which, as we saw earlier, still dominates the debate, it is not possible to prove the Fifth Postulate in a purely analytical or logical way, but it does seem to require recourse to an extralogical or intuitive element, since it refers to a concrete existence.

Vailati's first response discusses Brentano's observations in little detail and perhaps with a certain impatience, giving them a particular twist, transforming it into a discussion about what an axiom is and what its relationship is to what is deduced from it

> the passage from Aristotle contains expression so general and vague that it seems to me quite impossible to make any conjectures at all concerning the nature of the geometrical proofs to which it refers. As you rightly point out, we cannot even be sure of the meaning of the word (parallel) in that passage. The only thing certain is that Aristotle's observation does not refer to the reasoning involved in the indicated propositions in Euclid" (1982 p.8)

Diplomatically, Vailati suggests that it seems that the disagreements are merely verbal and that one can certainly say different things about parallels if they mean different things. As far as the logical status of the Postulate is concerned, both agree that it is non-analytic and thus indemonstrable in a purely logical way.

Not satisfied, Brentano then responds by trying to make his position explicit, and now the contrast between the two positions becomes clear. Brentano's objection is not that Euclid has added an axiom to his list which cannot be proved and is not immediately true, but that his version of the Fifth Postulate is not a “true axiom” (1982, p.9). Brentano's definition is as follows: “(...) a judgment that is made immediately obvious only by concepts” (1982, p. 9) (Brentano). However, it is not a question of understanding axioms here, as it might seem at first glance, as the equivalent of an analytical judgment in the Kantian sense, nor of empty tautologies, as in Mill (or Locke's “triffling propositions”). Although they coincide in their universality and necessity with the latter, they differ in terms of their informative character, or the expansion of our knowledge. Furthermore, when Brentano states the “obviousness” of such axioms, this is not an immediate or “at first sight” obviousness, so to speak, although these also define axioms, but rather that which results (“made obvious”) from a demonstrative process. The reader familiar with the final decades of the 19th century and the discussion of the reform of logic, which was the occasion for Brentano to express his views on the subject more fully, will be able to bring this closer to the notion of analyticity defended by Frege and will find several similarities that have yet to be studied in more depth. But, to recap briefly, for Frege, an analytic proposition is not one that is true because the meaning of the words or the predicate “is contained in the subject”, but analytic is everything that is derived by deduction from axioms and logical laws. Unlike Kant, for Frege (and Brentano), analytic judgments can be informative/expansive.

Now we can understand the point of divergence that made Brentano insist on returning to the subject: if the Vailati/Mill thesis were true, each and every informative proposition would have to be established by induction from experience. “We should say that our knowledge of the principle of contradiction is obtained by induction (*epagogé*)" (1982, p. 9).

But that's not the point. Rather, we must admit that: "For without perception, intuition, noticing, and distinguishing, we would have no concepts at all and therefore would be incapable of making judgments that are made evident by a concept" (1982, p. 9).

Neither intuition nor deduction can be limited or reduced to induction. On the contrary, there is also a type of analyticity of mathematical concepts that depends on experience and scientific practice, but is never reduced to them. The concept of intuition is no longer, as it was originally in Kant, direct and immediate contact with an object, but instead expands to also include the mediated result of a process, whose link with "intuition" (now in a properly Kantian sense) is the starting point, but not the limit for its application. Paraphrasing the famous saying about experience, and thus returning to the idea of Chisolm and Corrado's initial quote, we could say that in Brentano "intuition begins with experience, but does not end there". This idea, however, is not new, and is similar to neo-Kantian positions such as that of COHEN, expressed in the motto "Experience itself becomes concept, which must be constructed in intuition and thought" (COHEN, 1999, p. 104).

From this Brentano draws a series of conclusions. Even if there is recourse to intuition, mathematics, in contrast to the "inductive sciences - including mechanics" is "purely analytic in its character" (1982, p. 10). This is possible due to the expanded notion of analyticity and intuition adopted by Brentano, where the former can include the result of the latter without losing its analytic character. Furthermore, if only inductive propositions can be informative, then not even Mill could say that the Fifth Postulate is an axiom because "it is obvious that the proposition is totally incapable of direct confirmation by experience". At this point, Brentano turns to the geodesic measurements made by Gauss, which many have considered to empirically establish the truth of some kind of non-Euclidean geometry for physical space. Such measurements, even if large in relation to us, are insignificant in the face of the mathematical infinitude that is postulated in the discussion of parallels. This implies, therefore, that another type of justification of the 5th postulate is necessary if we don't want to fall prey to the "disease" of "sub-Euclidean geometries" (1982, p. 10) (i.e. elliptic) or "supra-Euclidean geometries" (1982, p. 10) (i.e. hyperbolic). Since geometry is analytic, and analytic truths cannot contradict each other, we must ask ourselves what makes Euclidean geometry true and the others false.

The theory that Brentano outlines works with the Kantian dichotomies, but alters their combinations, and can therefore work at the same time with the ideas of analyticity and intuition as both being contained in the justification of geometric knowledge: after concluding that geometry is analytic, Brentano states:

> The actual existence of such spatial points is directly verified in the spatial intuition of certain concrete spatial concepts. On a line, only two points coincide with each other. On a straight line, these two points are in opposite directions, corresponding to the opposite directions of their end points. Thus, straight lines have the same direction in all their parts (...). (1982, p. 10-11)

How can this be reconciled with what he had previously maintained?

> My answer is that we do not have to set this up as an axiom. We can demonstrate it on the basis of truths which are made immediately

> evident by the concept of a spatial continuum (and of every analogue of more developed coincidence relations). These truths were recognized, explicitly or implicitly, by Euclid himself. (1982, p. 11)

There thus seems to be a logical structure in space that is not synthetic in nature; although it needs “a concrete experience of space” to reveal itself.

After this lengthy response, with several statements in clear contrast to Vailati's empiricist and pragmatist tendencies, Vailati returns to the polemic more willingly, and the differences between their respective positions appear more clearly. Vailati begins by maintaining that, contrary to Brentano, neither “being immediately evident” nor “being able to be established by induction or direct confirmation” characterize axioms. In other words, rather than discussing axioms in terms of the metaphor of “sources of knowledge” (again, a common recourse for Brentano, Frege and Kant), Vailati prefers a discussion in terms of their application in science and knowledge, and more specifically their function within a logical system. For him, what defines whether something is an axiom is 1) its simplicity 2) its fertility. Axioms are axioms only in relation to a given deductive system, and we must choose them by criteria internal to the system itself, not external. If in Brentano the paradox was between analyticity and intuition, in Vailati it seems to be between his logicist tendency on the one hand and his conventionalist empiricism on the other.

In Vailati's own words:

> The purpose of axioms is simply to make possible the construction of a system of consequences (verified by experience) by means of a system of hypotheses which would be the simplest possible among those from which the

> same consequences would be deducible. To say that all of geometry is based upon the axioms is for me a pure metaphor and means only that all the propositions of geometry can be obtained by simple deduction (a series of syllogisms) from the chosen axioms; that does not in any way imply that those axioms ought to be, in themselves, more directly evident or more easily verifiable than all of the propositions which we deduce from them. (1982, 12)

And then, reaffirming his position: “Geometry differs from physics only in degree, not in nature” (1982, p. 12), which leads him to the following conclusion:

> In conclusion, one cannot give a precise answer to the question: Which are the true axioms of geometry? -- if I am right - - without first answering this question: What is the best way of ordering our knowledge of the properties of space, in such a way that they would appear as consequences of a limited number of fundamental hypotheses? And, of the various ways in which this can be achieved, some may be preferable in certain respects, and others in other respects (for example, some for the great evidence of the hypotheses, others, on the other hand, for their small number). (1982, p. 13)

Brentano's reply comes in an impatient tone: “I see that you are a follower of Avenarius and Mach, who think that general principles only help memory”. He returns to the Aristotle that had been the starting point of Heath's discussion and, through a recourse to the Aristotelian distinction between knowledge quoad se and cto. quoad nos, which he interprets in terms very close to a distinction between the context of discovery and the context of justification, to clear up Vailati's confusion.

After reproaching him for the indiscriminate use of "axioms" to refer to any kind of general proposition, Brentano explains that the difference between axioms and general propositions (such as empirical generalizations or laws of physics) is a difference between what is logical and primary and what is physical and derived:

> For geometers began with axioms without attempting to ground them in any way. But this would not be possible if they took the axioms in the way in which, for example, physicists understand the law of the conservation of energy. The justification of the law, as they conceive it, may require, not only a direct induction from an experience showing no exceptions, but often a chain of other complex thought processes. (1982, p. 15)

From there, the discussion moves on to the nature of the scientific method and the respective role of induction and deduction in discovery and justification, and in this respect Brentano concludes: "both mathematics and any inductive science would be impossible without recognizing the analytical (non-inductive) character of mathematics" (1982, p. 15). The subsequent discussion will not touch more directly on the issues that interest us at the moment, so we can now summarize some of the results.

6. Final considerations

Brentano certainly possessed a complex and nuanced theory of mathematical knowledge in general, and geometrical knowledge in particular, as this review has attempted to show. This theory, however, has not yet been adequately studied, and the study of Brentano's manuscripts will certainly give impetus, and material, to this endeavor. This theory fits directly into the mathematical-philosophical "mainstream" of the second half of the 19th century, where revolutionary mathematical progress and an epistemology of mathematics still strongly marked by Kant and his doctrine of the synthetic, a priori and intuitive nature of mathematics dialogued.

The discussion with Vailati and the other texts we examined show at least some theses that must be reconciled in the reconstruction of Brentano's philosophy of mathematics. The claim that the existence of parallel lines is something we grasp based on our intuition of the nature of space, while at the same time supporting the analytical character of mathematics, brings Brentano closer, as we said, to the neo-Kantians. Brentano, however, insists that what we intuit is not space (as Kant's "infinite given magnitude") but its nature, which only becomes explicit as we progress in our investigation. Intuition is therefore not "direct and immediate contact with a particular" in the Kantian sense, but an elaboration of the nature of space (for example by accepting parallel lines) on the basis of what is given. Thus, in Brentano, "intuition" functions as a source of knowledge (in a sense close to the neo-Kantians, such as Cohen and Natorp, and Frege) not as a sensible experience. Non-Euclidean geometries are excluded because 1) they do not coincide with our best possible scientific description of the world and 2) only one geometry can be true, since analytical truths cannot contradict each other. For Brentano, one of the "postulates" of pure geometry is that we have intuitions of a certain kind, which are the basis of mathematical knowledge, but this does not contradict the fact that pure mathematics and geometry consist of necessary propositions.

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Brentano and the notion of continuity[16]

1. Continua in the Ancient world

The notion of continuity has been present in the thought of philosophers and mathematicians at least since the time of the Ancient Greeks. This notion, however, is not an intrinsically clear one. In fact, different circumstances call for slightly different ideas regarding how to cash out this notion of continuity. Nowadays, in mathematics, the most common usage of this notion of continuity is perhaps in connection to functions. In this sense, a function – more specifically, let us think here of a simple function from real numbers to real numbers – is said to be continuous if one can draw its graph without taking the pen off the paper, i.e., if the line that we recognize as its graph does not contain any "jumps". The usual formal characterization of this property is the classical ε-δ property, which can be intuitively described here as the property according to which a "small" change of the argument of the function only produces a "small" change in its value at this argument — or, in other words, a change to a "nearby" argument point moves the value of the function also to a "nearby" value. We can see in this sense how this notion is also connected to the notion of "closeness". Nonetheless, this transition into a rigorous definition does away with a – perhaps original – intuition of gaplessness in the sense that we are now always considering a "close" argument point, but one that

[16] This chapter was written by Prof. Dr. Arthur Heller Britto and some of its parts were presented in more depth in the PhD dissertation "Brentanian continua and their boundaries" (Heller Britto, 2020a) and on the article of the same name in *Brentano Studien* v. 17 (Heller Britto, 2020b).

is indeed some distance away from the original point and that cannot be considered the "next" point.

Indeed, this notion of "gaplessness" is the guiding idea with which Ancient Greek philosophers considered what they called "continua". The first to formalize – if not to use this notion in an argument –[17] was Aristotle. His "official" definition was given in Book V of the *Physics*, together with the related more basic notions of "being next-in-succession" and of "contiguity":

> A thing is in succession when it is after the beginning in position or in form or in some other respect in which it is definitely so regarded, and when further there is nothing of the same kind as itself between it and that to which it is in succession [...].
>
> A thing that is in succession and touches is contiguous. The continuous is a subdivision of the contiguous: things are called continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. And in whatever way that which holds them together is one, so too will the whole be one, e.g. by a rivet or glue or contact or organic union.
>
> It is obvious that of these terms 'in succession' is primary; for that which touches is necessarily in succession, but not everything that is in

[17] For instance, Zeno surely relied on the infinite divisibility of space to formulate his arguments against motion.

> succession touches: and so succession is a property of things prior in definition, e.g. numbers, while contact is not. And if there is continuity there is necessarily contact, but if there is contact, that alone does not imply continuity; for the extremities of things may be together without necessarily being one; but they cannot be one without necessarily being together. So natural union is last in coming to be; for the extremities must necessarily come into contact if they are to be naturally united; but things that are in contact are not all naturally united, while where there is no contact clearly there is no natural union either. Hence, if as some say points and units have an independent existence of their own, it is impossible for the two to be identical; for points can touch while units can only be in succession. Moreover, there can always be something between points (for all lines are intermediate between points), whereas it is not necessary that there should be anything between units; for there is nothing between the numbers one and two. (*Physics*, Book V, 226b34-227a34)

We should pay attention to the fact that, in order to carry out the aforementioned distinction between "being next-in-succession", "being contiguous" and "being continuous", Aristotle introduces the notion of "limit". In particular, he is assuming that things that can be contiguous or continuous, i.e., what we would now call the domain of the non-discrete, all have these limits and that these limits can touch or even fuse into a single limit, or, as Aristotle puts it, they can "become one and the same". In order to standardize our terminology, we shall use the fairly synonymous term “boundary” for the greek word used by Aristotle, which is usually translated as “limit” or “extremity”. I would like to stress here that the translation into “limit” is by no means a bad translation. Indeed, it is usually the preferred word to carry out this translation in most scholarly contexts. However, our choice of terminology here, is simply to unify our discussion with the much later terminology found in Brentano and we should note that Aristotle talks about this notion of “limit” in a way that is parallel to the use of the term “boundary” in modern mathematics, as we can see in the following passage:

> We call a limit the extremity of each thing, i.e. the first thing outside of which no part [of the thing] is to be found, and the first thing inside of which every part [of the thing] is to be found. (*Metaphysics*, Delta, 1022a4–5)

Now, to understand this notion of “boundary”, it is important to go back to the notion of “an indivisible”. This term in Ancient Greece could mean different things — either extended atoms that cannot be divided because of some fundamental impossibility in its own nature, or as things that fail to be able to be divided because they lack extension in some spatial way.

This seems to be a fairly minor point, but, in fact, these two notions of an “indivisible” have extremely different logical implications for one's conception of them. Indeed, having some true extension, the first kind of indivisibles — that we might call something like “the thick conception of atoms” — can easily be thought of as composing an extended continuum; something like the Archimedean axiom would guarantee that, no matter how small the extension of these thick atoms, if we juxtapose enough of them, we will arrive at a continuum that is as large as we want. Therefore, although these thick atoms were fairly common place in the mind of Ancient atomists, it seems that Aristotle's notion of “an indivisible” was indeed the truly unextended version, as we shall justify more thoroughly further on in order to consider Aristotle's claim that continua cannot be made up of indivisibles.

Now, because of the goals we have set ourselves in this chapter, we shall didactically forget about extended atoms and define this concept of ``an indivisible” to encompass any geometrical figure that is unextended in at least one of the three spatial dimensions. Thus, a plane figure, a line, a point, these are all examples of such indivisibles.

Then, we should note as well that, although there is no clear explicit claim in Aristotle to identify these boundaries with indivisibles, it seems to be a very natural move to carry out this identification of the various possible boundaries of geometrical figures with one or another type of these indivisibles. For instance, it is extremely natural to consider two points as the boundary of a line segment, a circle as the boundary of a two-dimensional disc, a sphere as the boundary of a 3-dimensional massive ball etc.

Another important part of Aristotle's account of continua is, for the purposes of this chapter, the claim that boundaries are not substances. This is the case because this Aristotelian claim is closely related, as we shall see later, with Brentano's idea that a continuum cannot be thought of as composed out of indivisibles. But first, let us analyze how these notions are interconnected in Aristotle's account.

First, we should note that, in Aristotle's philosophical framework, everything that exists is essentially split into two classes of “beings”: the class of substances and the class of things that belong to those substances. Moreover, the main distinguishing property between these classes is that, whereas substances can have independent existence, i.e., they do not require the existence of anything else for their own existence, the things that belong to them can only exist on top of some specific underlying substance that is to be thought, therefore, as having an ontologically prior being or existence. This has consequences regarding how each of these different metaphysical entities can come into being and cease to be. Indeed, Aristotle says in the beginning of the last passage mentioned that

> besides what has been said, there are also paradoxes about coming into existence and ceasing to exist. It is thought that in the case of a substance, if it now exists without having existed previously, or later fails to exist after previously existing, it must be in process of coming into existence or ceasing to exist. But with regards to points, lines and surfaces, when they exist at one time without existing at another, they cannot be in the process of coming into existence or ceasing to exist. For as soon as bodies have been put together, one boundary does not exist, but has ceased to exist, and when they have been divided, the boundaries exist which did not exist before (for the point, being indivisible, was not divided into two). And if the boundaries are in process of coming into existence or ceasing to exist, from what are they coming into existence?
>
> It is similar with the now in time; for this too cannot be in the process of coming into existence or ceasing to exist, and yet it is thought to be ever different, which shows that it is not a substance. Clearly it is the same with points,

> lines and planes, for the same account holds, since all alike are boundaries or divisions. (*Metaphysics*, 1002a28-b11)

This passage portrays very clearly how, for Aristotle, boundaries are *sui generis* objects, that, for instance, can come to be without ever being in the processes of becoming and, conversely, can cease to exist without ever being in the process of ceasing to exist. Indeed, in the *Physics* he restricts the general metaphysical claim that nothing can exist (not exist) without being previously in the process of coming into existence (ceasing to exist) explicitly to *continuous* or *divisible* things:

> Hence it is apparent that what has come into existence must previously have been in process of coming into existence [...] in the case of things which are divisible and continuous. (237b10)

Note that all of these boundaries we have been considering are unextended in at least one direction and, thus, are all indivisibles. Hence, we shall here make our first exegetical assumption and suppose that the boundaries that Aristotle talks about are indeed these indivisibles. There is an interesting passage in book B of the *Metaphysics* that attests to this interpretation. In it, Aristotle says:

> if it is a magnitude, it is corporeal; for the corporeal has being in every dimension, while the other objects of mathematics, e.g. a plane or a line, added in one way will increase what they are added to, but in another way will not do so, and a point or a unit does so in no way. (1001b10-11)

The connection seems to be between having extension in some spatial dimension and being able to increase something's size “in one way or another”. More specifically, we claim that these “ways” in which something might add to something else are precisely what we would recognize now as spatial dimensions. And right after that, Aristotle claims that adding such indivisibles would surely increase the “number”, but not the “size”, for how can

> a magnitude proceed from one such indivisible or from many? It is like saying that the line is made out of points. (*Ibid*., 1001b17-19)

What seems to be claimed here is that, although surely boundaries seem to play a vital role in the Aristotelian characterization of continua,[18] they are *not* assumed to compose these continua as their building blocks. That is clear, for instance, in the following passage, that opens the sixth book of the *Physics*:

> Now if the terms ‘continuous’, ‘in contact’, and ‘in succession’ are understood as defined above—things being continuous if their extremities are one, in contact if their extremities are together, and in succession if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles: e.g. a line cannot be

[18] Indeed, we find Aristotle characterizing the notion of a "body" by means of this notion of "boundary":

If ‘bounded by a surface’ is the definition of body there cannot be an infinite body either intelligible or sensible. (*Physics*, 204b5) composed of points, the line being continuous and the point indivisible. For the extremities of two points can neither be one (since of an indivisible there can be no extremity as distinct from some other part) nor together (since that which has no parts can have no extremity, the extremity and the thing of which it is the extremity being distinct).

Moreover, if that which is continuous is composed of points, these points must be either continuous or in contact with one another: and the same reasoning applies in the case of all indivisibles. Now for the reason given above they cannot be continuous; and one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous; for that which is continuous has distinct parts, and these parts into which it is divisible are different in this way, i.e. spatially separate.

Nor, again, can a point be in succession to a point or a now to a now in such a way that length can be composed of points or time of nows; for things are in succession if there is nothing of their own kind intermediate between them, whereas intermediate between points there is always a line and between nows a period of time.

> Again, they could be divided into indivisibles, since each is divisible into the parts of which it is composed. But, as we saw, no continuous thing is divisible into things without parts. Nor can there be anything of any other kind between; for it would be either indivisible or divisible, and if it is divisible, divisible either into indivisibles or into divisibles that are always divisible, in which case it is continuous.
>
> Moreover, it is plain that everything continuous is divisible into divisibles that are always divisible; for if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact. (*Physics*, 231a18-231b17)

This passage has become fairly famous and it essentially deduces contradictions from the assumption that points can be either next-in-succession, in contact or continuous with each other. The main result from this argument, however, is that continua cannot be mere aggregates of indivisibles – and this fact is indeed brought back by Aristotle in the last paragraph to lend its weight to the original characterization of continua as infinitely divisible. However, this position had already been explicitly made in the *Physics*, albeit not in its present fully abstract form. For instance, let us consider the following passage:

> the ‘now’ is not a part: a part is a measure of the whole, which must be made up of parts. Time, on the other hand, is not held to be made up of ‘nows’. (218a6-8)

This passage seems to raise the reader's attention to the “indivisible” character of what Aristotle calls “the now”. This is done, however, by means of another notion, that of “measurability”. Usually, in Ancient Greek texts one sees this word used in relation to the mathematical distinction between rational and irrational magnitudes, i.e., between magnitudes that share a common unit of measurement and magnitudes that do not.[19] Here, however, we have a slightly different, albeit related meaning being used. Indeed, here the question is not whether two magnitudes share a common unit and, therefore, are rational; the question here is whether some alleged part of a magnitude (*viz*. an instant as a part of a time span) can be superimposed on the magnitude a natural number of times so that at the end of the superimposition, the whole original magnitude is covered. This seems to be the way the notion of measurement acts in this context and Aristotle seems to be equating the notion of “a part of a continuous magnitude” to the notion of “being able to be superimposed on the original magnitude a natural number of times so that at the end of the superimposition, the whole original magnitude is covered”. In this respect, then, he concludes that the instant, or the “now”, is *not* such a part. This conclusion, however, bears a strong indication that Aristotle is indeed thinking about the “now” essentially as what in our terminology has been called “an indivisible” or “a boundary”.

Therefore, after all this I believe we can be confident in our interpretation of boundaries as indivisibles and in our reading of Aristotle's position as being that boundaries are not parts of continua and, therefore, that the latter

[19] E.g., we can think about a pair of lines with 2m and 3m and a pair of lines with 2m and $\sqrt{2}$m. In the first case, there is another magnitude, say, a line with 1m, that can be superimposed onto the original lines a natural number of times (2 and 3 times, respectively). However, in the second, there is no such magnitude.

cannot be composed of the former. Indeed, here is Aquinas' conclusion regarding this topic in his *Commentary to Aristotle's* Metaphysics:

> And the truth of the matter is that mathematical entities of this kind are not substances of things [and thus cannot compose things], but are accidents which accrue to substances. But this mistake about continuous quantities is due to the fact that no distinction is made between the sort of body which belongs to the genus of substance and the sort which belongs to the genus of quantity. For body belongs to the genus of substance according as it is composed of matter and form; and dimensions are a natural consequence of these in corporeal matter. But dimensions themselves belong to the genus of quantity, and are not substances but accidents whose subject is a body composed of matter and form. (pp. 189-190)

And with these things in mind, we can now conclude that, according to Aristotle, the notion of “a continuous thing” is characterized both by its infinite (potential) divisibility and by the fact that it possesses a fundamental unity that is characterized by the property that any pair of parts which exhaust the continuum must share at least a boundary. Note, however, how both properties eventually boil down to the assumption of these indivisible boundaries that are (potentially, perhaps) everywhere in the continuum, since everywhere in it is a possible place of division and since everywhere in it can be conceived of as being a place where two parts of the continuum are actually fused together – a condition that is also determinant for the Aristotelian conception.

Thus, we see how these two properties of continua are, for Aristotle, merely flip-sides of the same theoretical coin, which is the assumption of this close relation between continua and boundaries. However, what is also interesting for our purposes in this chapter is to note a sort of converse of this claim that any two parts of a continuum must share a common boundary — *viz*. the fact that a pair of things that merely touch at their boundaries and, thus, are not fused together into a single continuum, must each have its own boundary which is somehow collocated with the boundary of its corresponding contiguous counterpart. Indeed, this independently follows from the fact that a "thing" or a "body", as we saw, must have a boundary, together with the fact that two things, if they are to *fail* to merge into a single continuous entity, their touching boundaries cannot become the same. In this respect, we have the following passage from Aristotle:

> In the act of dividing the continuous distance into two halves one point is treated as two, since we make it a beginning and an end; and this same result is produced by the act of counting halves as well as by the act of dividing into halves. But if divisions are made in this way, neither the distance nor the motion will be continuous; for motion if it is to be continuous must relate to what is continuous; and though what is continuous contains an infinite number of halves, they are not actual but potential halves. If he makes the halves actual, he will get not a continuous but an intermittent motion. (*Physics*, VIII, 263b1-263b6)

2. Development of the Aristotelian conception in the Middle Ages

The beginning of the Middle Ages was marked by an intellectual appropriation of neo-Platonic ideas into a Christian theological worldview. This much is certain. However, this was also accompanied by a lack of interest in Aristotelian ideas regarding continua. What survived the end of Antiquity was more a discussion regarding the possibility of leaps and regarding the paradoxes of motion than a thorough appropriation of Aristotle's abstract account of continua and their boundaries.

However, as Aristotle's works found their way into the newly established universities of Europe, his ideas on this topic quickly became again fairly canonical. More precisely, much like most other topics touched by the Stagirite philosopher, the discussion regarding continua and indivisibles in the Late Middle Ages was heavily constrained by the account we have been analyzing in this chapter.

Of course, as in other areas of inquiry, Late Medieval thinkers expanded upon Aristotle's original account, but most of those thinkers carried out all kinds of logical and semantic maneuvers to position their accounts as close to the original Aristotelian account as possible.

This is not the place for a complete discussion of the myriad of paradoxes and their multiple solutions by Scholastic philosophers, but what is of interest for us here is a particular general account on the moment of transition that provided the Aristotelian account with tools to attack these paradoxes in a way that is surely a logical ancestor of Brentano's approach in the 20th century, in the sense that it hinges on a particular approach to the metaphysics of boundaries that is, from a logico-formal perspective wholly analogous to Brentano's. However, a different question is whether these ideas were indeed a truly historical ancestor to Brentano's. This is much harder to say, since we cannot be sure what exactly in the Aristotelian tradition Breentano actually had access to. However, there is such a logical agreement between this tradition and Bretano's ideas that one must surely wonder whether this is where Brentano got his ideas about boundaries from.

We are referring to the tradition that arose in the 14th century with the work of some philosophers – such as Henry of Ghent, Hugh of Newcastle, John Baconthorpe and Landulf Caraccioli[20] – who began to take seriously a certain possible attack on what was by then the “orthodox” Aristotelian notion of continua. The problem they raised attention to has to do specifically with the application of Aristotle's general theory of continua to the particular case of change, or, more specifically, motion – which is indeed a particular case of the former broad notion of change in general, as it is, according to Aristotle, simply a change in location.

A certain change of some substance is a process in which it either has at some time t0 some condition[21] which it fails to have at some later time t1 or vice-versa. Now, if both the moment in which the substance stops having the condition and the one in which it starts not having it coincide and, since something has some condition up to, and including, the last moment of the condition's presence and does not have it from the moment it stops having it onwards, then it seems that we must conclude that this moment of transition is a moment in which the substance both has and does not have the condition in question – something

[20] On this topic, the papers Knuuttila and Lehtinen (1979), Kretzmann (1982) and Spade (1982) are particularly interesting and thorough. Also, cf. the discussion between Sorabji (1976) and Kretzmann (1976b).

[21] Nowadays, perhaps the word "property" might be more appropriate here, but we shall stick to the historical vocabulary.

that would blatantly contradict Aristotle's principle of contradictories.

Now, to solve this problem, the so-called quasi-Aristotelian Medieval philosophers introduced in connection to it what Kretzmann (1982) calls “the divided instant” (p. 276), which was originally characterized in the following passage of Baconhtorpe:

> The termini of a change are separated from each other only as much as the duration of the change that mediates between the termini, but an instantaneous change does not endure except for an instant alone; therefore its termini are separated not in accordance with with the parts of a duration, but solely in accordance with the order of nature. (*Commentary on the “Sentences”*, L. III, d. 3, q. 2, art. 3)

According to him, the usual interpretation of his contemporaries makes it so that

> [...] something false is imposed on the Philosopher. For the Philosopher there does not save the contradiction between being and not-being in that way [...]; instead, the Philosopher saves the contradiction in this way, that the instant is divided into a beginning and an end in such a way that the instant's first sign, which corresponds to the terminus a quo of an instantaneous change, measures the *ultimum* of the not-being, and its last sign measures the *primum* of the being [...] (*Commentary on the “Sentences”*, L. III, d. 3, q. 2, art. 3)

The gist of the idea here is that, if one can think about the instant of transition as being composed of two distinct instants, then one is free to locate each contradictory condition in one of the composing instants and thus to reject the conclusion that these contradictory conditions must be both present in the same instant, which is the conclusion that seems to be the main problem for the quasi-Aristotelian thinkers in the original Aristotelian framework.

The whole talk of “instants of nature” seems to be an artifice found by these philosophers to introduce such a division in the instant of change, which is what does the main logical labor in this discussion. It is not the fact that these new instants are “of nature” that allows the quasi-Aristotelians to do away with the original problem, but the simple fact that the assumption of these moments of nature allows one to think about the original instant of change as somehow being *divided*. In this sense, then, Kretzmann's characterization is fairly on target, for it calls one's attention exactly for this divided nature of the instant of transition in the quasi-Aristotelian tradition.

That this idea is to be found originally in Aristotle was surely claimed by the so-called pseudo-Aristotelian proponents. However, it is not so far-fetched to believe that the Stagirite did have, perhaps not such a fully detailed account, but an idea that the point of transition did somehow pertain both to the former and latter condition. Indeed, he says that

> [i]t is also plain that unless we hold that the point of time that divides earlier from later always belongs only to the later so far as the thing is concerned, we shall be involved in the consequence that the same thing at the same moment is and is not, and that a thing is not at the moment when it has become. It is true that the point is common to both times, the earlier as well as the later, and that, while numerically one and the same, it is not so in definition, being

> the end of the one and the beginning of the other; but so far as the thing is concerned it always belongs to the later affection. (263b9-263b12)

Moreover, the interesting point to be made here, in the context of the broader discussion of this chapter, is that this divided nature of the moment of transition, as it is conceived in the quasi-Aristotelian tradition, is perhaps what will allow the Aristotelian-inspired Brentano in the 20th century to talk about a multiplicity of indivisible boundaries being collocated in order to elucidate cases in which different continua meet. In particular, we must note the protagonism that the problem regarding the moment of transition plays in both contexts and, although this moment of transition is, for Brentano, a much more abstract notion – not being necessarily restricted to any kind of physical motion or change –, which encompasses boundary points in which, e.g., differently coloured continua touch, we must attest to the fact that the recognition of a multiplicity of collocated boundary points plays the same logical role in the Brentanian discussion and eventual solution to the problem as the divided instant in the quasi-Aristotelian solution to the paradoxes of change. Thus, this Brentanian solution too, notwithstanding its being more abstract and less couched in obscure scholastic terms such as "instants of nature", will have, in very broad terms, the distinction between different indivisibles that are present at the same "point" or "instant" as its essential logical structure.

3. The manifold-theoretic view on continua

However, before we talk about the Brentanian solution, we have to mention the mathematical revolution that happened in the late 19th century. Nowadays, it thoroughly accepted that any mathematical theory, from abstract algebraic group theory to differential geometry or functional analysis, can be and, indeed, is perhaps supposed to be – if one is to consider it as being actually thoroughly formalized so as to achieve the modern level of logical rigor – recast inside formal set theory, say, as a collection of sentences that follow logically from the ZFC axioms. Even category theory, which is often thought to be a new – and better, since more in line with many mathematical customs and intuitions – logical foundation for modern mathematics, is hardly ever introduced without recourse to set theoretical notions. However, this omnipresence of set theoretical concepts and methods is a late 19th century creation, whose consolidation occurred well into the first half of the 20th century with the introduction of the formal axiomatic systems we know today as ZFC or BGvN and with its use in the development of many areas of mathematics, but mainly of point-set topology.

Now, this moment of consolidation of these new mathematical methods is precisely when Brentano is writing about his ideas on continua. Hence, it is only reasonable to assume – and, indeed, fairly clear from the few mathematicians and mathematical notions he mentions in his essays – that, in doing so, he is replying to the ideas that have been around in the mathematical discussions regarding continua and that his thoughts must be understood in the context of the different mathematical methods for defining the continuum of real numbers which were just 30 or 40 years old at the time he was struggling with his own ideas on the subject.

It is very unlikely that Brentano himself had a thorough understanding of the whole mathematical literature that was devoted to these notions in the latter part of the 19th century and early 20th century and it is almost surely the case that he did not have access, for instance, to Hausdorff's textbook consolidating the new topological ideas, as came out in the very year as the former was dictating his essay on continua. There are even passages in this essay in which it seems clear that he misunderstands the relevant mathematical ideas – in particular the distinction between different sizes of infinity. However, through the work of the mathematicians whose work he was acquainted with – in particular, Riemann and Poincaré are the ones he mentions in this respect –, Brentano was definitely exposed to the core of mathematical ideas that constituted the manifold-theoretic view of continua and he is surely engaged with this new tradition when he expounds his own ideas on the subject.

But in what consists this “manifold-theoretic” view of continua? Well, essentially, during the Middle Ages, there was the introduction and gradual acceptance of integration methods that assumed that some given continuum was indeed composed out of other continua of less dimensions, which, in our characterization with respect to the original continuum, were thus indivisible. These methods became mainstream with time and, therefore, the assumptions it required became more and more palatable to mathematicians.

Eventually, sometime between the 19th and 20th centuries, mathematicians established a rigorous foundation of mathematics that relied heavily on indivisibles. We are talking about set theory and its application in point-set topology. In this conception, continua[22] were definitely composed out of indivisibles – indeed, out of dimensionless points. Thus, when Brentano was writing, the situation was such that the Aristotelian assumption about continua was turned on its head and the new mathematical orthodoxy was

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[22] Or, indeed, *the* continuum, as it became common to understand what was termed "continua" as some subset of some given cartesian product of the real numbers – which was now considered "the continuum".

very much leaning towards the acceptance of indivisibles as the constituents of continua.

4. Brentano's conception of continua

Although, in the decades that preceded the turn of the 20th century, the Aristotelian position – according to which continua cannot be composed out of indivisibles – seemed to be completely dead and buried by the newly established mathematical orthodoxy that, as we mentioned, was founded upon the account of continua in terms of the newly developed set-theoretic topological concept of a continuuous manifold, we shall see that Brentano had the courage to go against the establishment and to propose a new account that has deep roots in the original Aristotelian tradition, but that builds on it in order to create a cohesive and credible picture of the ontology of continua and their – as Brentano will think about them – inseparable boundaries.

Brentano's main philosophical goal was to provide the philosophical foundations for a truly scientific psychology, i.e., for a rigorous study of the subjective dimension of reality, which was delimited by him as that portion of reality that is characterized by the property he called intentionality. In this context, for him, a philosophical account of certain objects was to be carried out as an account of how these objects are intentionally presented or, in a more Brentanian terminology, present to the cognizant subject.

Thus, it is no wonder that, when it comes to the special kind of objects that we're calling continua here, we have Brentano claiming that

> it is much rather the case that every single one of our intuitions — both those of outer perception as also their accompaniments in inner perception, and therefore also those of memory —

bring to appearance what is continuous. Thus in seeing we have as object something that is extended in length and breadth which at the same time shows itself clearly as allowing us to distinguish a front and rear side and thus as characterized as the two-dimensional boundary of something extended in three dimensions. And since this continuous something presents itself to us who see as being our primary object, we see also at the same time and as it were incidentally, our seeing itself, that is, we are conscious of ourselves as ones who see, and we find that to every part of the seen corporeal surface there corresponds a part of our seeing, so that we also, as seeing subjects, appear to ourselves as something continuously manifold. And still more, what appears to us first and foremost is rest and motion; so also persistence and gradual change appear to us as primary qualitative objects. This happens in that, whilst certainly in our perceptual presentation of the primary object we are never able to present the same place filled with two qualities simultaneously, still we are able to present it as filled with one quality as present, with another as most recently past, and with yet another as further past, whereby the transition from present to further past takes place in an entirely continuous manner. Thus once more we appear to ourselves, in seeing phenomenal qualities following each other in a temporally continuous way or in seeing them persisting continuously in time, as something that is continuously manifold. (1988, pp. 4-5)

Here, we have a passage in which Brentano talks about how this notion of continuity is present both in pretty much all our intentional objects of experience and indeed in our very temporal nature. Therefore, in accord with this assumption, his project is to present an account of this property of continuity as it is ubiquitously present in our common experience of both the objects in the external world and of ourselves as subjective observers of this world. However, in doing so, Brentano ends up arriving at an idea of continuity that is in sharp disagreement with the established mathematical reconstruction of this concept that we termed the “manifold-theoretic” account.

Brentano's criticism of the mathematical constructions of the continuum that were being carried out in the late 19th century by figures like Cantor and Dedekind are intimately related to a distinction regarding the two ways through which one can acquire concepts in general, and in particular the concept of a continuum. According to Brentano's anti-rationalist account, there cannot be any *a priori* concepts. Thus, any concept, according to him, is either given straight through some intuition or is constructed by means of some logical components which were usually called marks (*Merkmale*) in the German epistemological tradition dating back at least to Kant. It is Brentano's view that the mathematical constructions of continua – essentially of the continuum of real numbers – are examples of such second way of obtaining concepts, so that his criticism of such constructions is fundamentally connected to a criticism of the view according to which the notion of a continuum can and must be obtained by such a logical construction and, therefore, is equivalent to a justification of his starting point according to which continua are abundantly given to us in experience, so that an account of the notion of a continuum must necessarily be obtained in these particular intuitions we have of individual continua, or in Brentano's own words,

> the concept of the continuous is acquired not through combinations of marks taken from different intuitions and experiences, but through abstraction from unitary intuitions. (1988, p. 4)

Brentano's criticism does not, however, really amount to a thorough refutation. Its form is much better understood as a two-pronged attack: first, he hints towards some essential kind of “intuitive unnaturalness” of all the mathematical constructions of continua that were founded on what we're calling the “manifold conception” of continua; and, then, he shows how there is an alternative way of thinking about the various kinds of continua which are presented in intuition that is much more in accord with some natural assumptions about them. This alternative way, however, is essentially not new, but springs from Aristotle's conception of continua, which is itself based on the Greek philosopher's thorough denial of actual infinities and his intuition that the notion of a continuum should be intimately connected with the notion of its boundary or its limit, which is in its turn something whose being is derivative or dependent on the being of the continuum it bounds.

Under these assumptions, it would be an absurdity to attempt at a construction of some continuum by starting from its lowest-dimensional boundaries, *viz*. its points. Such a construction would amount to nothing other than a blunt metaphysical putting of the cart in front of the horses, in that it would amount to a construction of a certain entity out of other entities whose being would be highly dependent on the first entity's being to start with. In this respect, we have the following illuminating passage:

> If something continuous is a mere boundary then it can never exist except in connection with other boundaries and except as belonging to a

> continuum which possesses a larger number of dimensions. Indeed this must be said of all boundaries, including those which possess no dimensions at all such as spatial points and moments of time and movement: a cutting free from everything that is continuous is for them absolutely impossible. And this allows us to grasp very clearly the topsy-turvy character of the above-mentioned attempt at construction of the concept of the continuous through interpolation of fractional numbers, where every fraction is supposed to have existence without belonging to a series of fractions. (1988, p. 7)

In the criticisms of Brentano we can distinguish two key points that serve as foundations for the whole argument, *viz*. the question regarding the ontological status of boundaries that we mentioned above and the question regarding the possibility of actual infinities. The question whether actual infinities are metaphysically possible is one that dates back at least to Aristotle – who answered strongly in the negative – and Brentano seems to be following the Ancient Greek philosopher closely in this regard when he denies that the mathematical constructions could ground our notion of continuity, for these constructions would require an actual infinity of interposed elements in any given continuous extension.

The acceptance of actual infinities seems to go, after the mid 19th century, hand in hand with the new set-theoretic foundation of modern mathematics. Indeed, since the groundbreaking work of Cantor, that arguably established set theory as an acceptable mathematical theory, we have the establishment of different transfinite cardinalities as a mathematical fact and the study of their arithmetical and geometrical properties as part of the set-theoretical work to be done. In particular, we have, in the context of the discovery of different sizes of infinity, the revolutionary distinction between density and continuity, which is the distinction that somehow grounds the numeric distinction between the merely rational numbers and the truly continuous set of real numbers.

Before the work of those mathematicians that aimed to establish a precise formulation of this property of continuity, which distinguished the real numbers from the merely rational numbers, the notion of "continuity" seemed to be related with the property of the real numbers – or, indeed, of any continuous extension – according to which, between any two real numbers, no matter how close together, we can always find another real number between them. This property is nowadays unambiguously called "density" and it is sharply distinguished from what we call "continuity", which is characterized by a number of equivalent assertions, *viz*. the existence of cuts or of lowest upper bounds for bounded subsets etc.

Unfortunately, even though Brentano was by no means a complete stranger to the latest mathematical developments of his time, he nonetheless surely failed to capture their full meaning. In particular, he never clearly understood this distinction between density and continuity, and continued to think about the latter in terms of its older characterization in terms of properties that resembled more the modern notion of density than the proper modern notion of continuity per se. I believe that this lack of proper understanding on Brentano's part unfortunately prevented him from engaging more thoroughly in the discussion regarding actual infinities, in the sense that any of the continuity properties requires a more robust acceptance of actual infinities than any merely dense set does, since one easy way of understanding what it is to be continuous certainly would require the property of being composed by a continuously or non-denumerably infinite number of points, although this requirement would by no means be sufficient since, e.g., the

Cantor set is non-denumerable, but one would hardly say that it is continuous, for, as a subset of the real numbers, considered with their usual topology, the Cantor set is *not* connected and, from a measure theoretical point of view it has measure zero.

However, what seems to be the case is that Brentano believed that his account of boundaries as dependent entities would be more in synchrony with Aristotle's tradition of denying actual infinities. This is because his account of boundaries as dependent entities would equate them with universals, which according to him do not have a proper kind of existence and could correspond to many different individuals. Indeed, he says that

> [b]ecause a boundary, even when itself continuous, can never exist except as belonging to something continuous of more dimensions (indeed receives its fully determinate and exactly specific character only through the manner of this belongingness), it is, considered for itself, nothing other than a universal, to which — as to other universals — more than one thing can correspond. (1988, p. 8)

This move to consider boundaries as universals has the upshot of allowing him to consider the inner boundaries of some extended continuum as merely potential, i.e., as not being actually instantiated by a certain individual, whereas the outer boundaries would be actually instantiated. Thus, one might be able to hold the view that only the outer boundaries have actual existence and, therefore, that the number of things with actual existence remains finite.

Before we get to the topic of coincidence of boundaries, however, we still need to discuss the second background assumption in Brentano's criticism, which is that there is an intrinsic relation between a continuum and its boundaries, according to which the reality of the latter is strictly speaking dependent on the reality of the continuum itself. In other words, for Brentano, boundaries – as, indeed, any other universal – can have no independent existence and, therefore, can only exist as boundaries of a certain particular higher-dimensional continuum.

Beside the passages in the compendium about space, time and the continuum that we mentioned above, we find clear expressions of this thesis in Brentano's *Theory of categories*:

> no continuum can be built up by adding one individual point to another. And a point exists only in so far as it belongs to what is continuous; points may be joined together just to the extent that they do belong to the same continuum. But no point can *be* anything detached from the continuum; indeed, no point can be thought of apart from a continuum. (1981, p. 20)

In the special context of Brentano's aforementioned intentional – or one might say “proto-phenomenological” – approach to the ontology of the objects which are presented to us, what we have is that the only self-standing continua, besides the one-dimensional temporal continuum, are essentially the three-dimensional bodies of outer experience; all other lower-dimensional continua are to be thought of as boundaries of some three-dimensional body.[23]

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[23] Or, of course, boundaries of boundaries of such bodies, for the case of one-dimensional boundaries, and boundaries of boundaries of boundaries of such bodies, for the case of points. In this sense, it is useful to mention the following clear passage from Brentano (1988):

The first, *viz*. the temporal continuum and the three-dimensional bodies of outer experience, are what Brentano calls “primary continua” in the sense that more dependent continua have their existence founded upon the – logically, as opposed to chronologically – previous existence of these primary continua. For example, besides the boundaries themselves, one might mention as secondary or dependent continua any kind of property, such as color or hardness, that is present in some primary continuum. Thus, a red parallelepiped must be understood as being composed of the primary continuum that constitutes the parallelepiped's volume and which is both bounded by the six rectangular faces which constitute its outer boundary and filled by all the rectangular inner boundaries that can be transformed into outer boundaries of its parts, were the parallelepiped to be

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> If something continuous is a mere boundary then it can never exist except in connection with other boundaries and except as belonging to a continuum which possesses a larger number of dimensions. (p. 7)

However, this account is not something he developed late in his life, but was indeed a point that stayed fairly unchanged, as we can attest from this earlier passage from the *Descriptive Psychology* manuscript:

> It is to be noted in this context that the one- and two-dimensional ones, like points, are only possible as boundaries, by themselves they are nothing. Everything they are, they are only in connection with the third dimension, i.e. with the physically spatial. We said earlier that a spatial point never exists without a continuum. This must still be more precisely determined to the effect that it can never exist without connection to three-dimensional spaces. (1982, p. 120)

divided; but also, it has a secondary continuum in its composition as well, which is to be identified with the red color that permeates its outer boundary and is to be regarded, according to Brentano, as something continuous, since it is just as extended as the outer boundary itself. Indeed, he says that

> the colour, too, appears to be extended with the spatial surface, whether it manifests no specific colour-differences of its own — as in the case of a red colour which fills out a surface uniformly — or whether it varies in its colouring — perhaps in the manner of a rectangle which begins on one side with red and ends on the other side with blue, progressing uniformly through all colour-differences from violet to pure blue in between. In both cases we have to do with a multiple continuum, and it is the spatial continuum which appears thereby as primary, the colour-continuum as secondary. (1988, p. 15)

We would like to stress here that, although our example considered a uniform color as a secondary continuum, it is clear from Brentano's passage, that he also thinks of a continuously varying colour as a possible example of a secondary continuum.

Now, this talk about dependent boundaries and secondary continua is, indeed, the main point of Brentano's account. They are the most distinctive characteristics of this account and they are the ones that most strongly relate it to the Aristotelian tradition. Moreover, their recognition brings about the possibility of describing two new and interesting properties of continua. These are the notions of *plerosis* and *teleosis*, which are intuitively to be understood, respectively, as a measure of the "fullness" of a certain boundary and a measure of the "degree of change" of a certain secondary continuum.

On the other hand, this subsumption of the boundaries of continua to the class of universals also allows Brentano to think about the coincidence of boundaries. An important part of Brentano's account is that this notion of "limiting some higher-dimensional continuum" does not have to be – and moreover usually is *not* –, according to him, total. Much more commonly, boundaries only bound other higher-dimensional continua in a restricted portion of the possible total number of directions that are present in the higher-dimensional space in which the bounded continuum is embedded. For instance, according to the Brentanian account, the disc that bounds the northern hemisphere of a solid sphere only does so in the north-pointing direction, but not in the south-pointing direction. Now, to make this point clearer, Brentano introduces a concept that is supposed to be a function of the number of directions in which a given boundary bounds a higher-dimensional continuum in relation to something like "the total number of directions in which the boundary could bound some higher-dimensional continuum", i.e., some kind of "measure" of the degree to which the boundary in question actually fulfills the possibility of being a boundary in every possible direction of the space that embeds the higher-dimensional continuum the boundary is a boundary of.

What we have here is an intuitive proposal of a concept that can have a much deeper mathematical significance. The background idea here is certainly something like the Jordan curve theorem, which states that any closed Jordan curve or, in other words, any closed non-self-intersecting curve on the plane divides the plane into two connected regions. Thus, one can think about this Jordan curve as a boundary of either of these two regions, or of both. In the former case, we shall say that the curve has "half plerosis" and in the latter that the curve has "full plerosis". Indeed, this theorem – which, by the way, was the center of much discussion during the turn of the 20th century – deals with the plane as the background space, which is certainly one dimension less than what Brentano considers to be the embedding space for the “usual continua of outer experience”. However, by the time Brentanto is dictating his notes, Brouwer and Lebesgue have already used homology theory to prove a generalization of the Jordan curve theorem to higher dimensions.

Thus, because of these theorems, the notion of the plerosis of boundaries that have one dimension less than the embedding space is very simple and amounts essentially —- in the case of boundaries which are images of an injective continuous mapping from a sphere — to the statement that this boundary has full plerosis if the portions of the embedding space into which it is divided by the boundary are not actually split up by the boundary, so that the boundary is not an actual outer boundary, but simply an inner boundary,[24] or half plerosis otherwise.

However, the situation is more complicated when one tries to generalize this idea to boundaries with smaller dimension when compared to the embedding space. This is because, for instance, given a line embedded into three dimensional space, this line can be a boundary of an infinite number of different half planes, so that ascribing to it a plerosis with the formula

$$P=1/n, \qquad (*)$$

[24] In the sense of a possible place in which the total object can be divided into two parts. In this context, Brentano says that

> [w]here we have to do with the interior of a continuum, every point has full plerosis, i.e. is connected in every conceivable direction with the relevant continuum. (1988, p.20)

where *n* is the number of portions of the embedding space of which the boundary in question can be a boundary of, does not make any mathematical sense.

If the situation is such that, some higher dimensional continuum is, as a matter of fact, partitioned into a finite number of symmetric regions that meet at a single boundary, then one can surely ascribe a plerosis to this boundary with (*). A simple example to portray this situation is a disc without one of its quadrants.

The center point of the amputated disc is a boundary of each of the remaining three quadrants, but certainly not of the missing quadrant, so that we could ascribe to this point a plerosis of 3/4.[25]

The more general case in which the higher dimensional continuum is *not* partitioned into a finite number of symmetric regions that meet at some given boundary, could be studied with the help of measure theory.

Now, with this in mind, we can shift our attention to the most interesting consequence of this notion of plerosis, which is that it enables one to give a new and surprisingly down to earth account of what it is for two continua to *touch* each other. Under the background assumptions of the set-theoretic conception of continua, the notion of two bodies touching is a little abstract and arbitrary. This is because in this context, for two bodies to touch, one of them would

[25] In this discussion, for the sake of clarity and simplicity, we are assuming the disc to be the embedding space. Otherwise, i.e., if we were considering the more realistic case of a disc embedded into the real 3-dimensional space, then we would need to consider directions that are not co-planar with the disc as well, so that we would not end up with a finite partition of the possible regions the center of the disc could be a boundary of and, therefore, could not ascribe to this point a plerosis according to (*).

have to be “open”, *i.e.*, one of them must not contain its own boundary, so that the actual boundary of one continua could be located where the boundary of the other continua would be if it existed. But this situation is simply a more general statement of the position which was famously termed a “monstrous doctrine” by Brentano.[26] The reason for doing so was that he could not accept the symmetry of this situation and the analogous situation in which the continua were flipped and the open one is now considered closed and vice-versa.

On Brentano's account, though, there is a very simple and intuitive definition of contact that hinges on his definition of plerosis. Since not all boundaries have full plerosis, we can think about the possibility of *coincidence* of

[26] Here is Brentano's famous passage against Bolzano's "monstrous doctrine":

> According to the doctrine here considered, in contrast, the divisions of the line would not occur in points, but in some absurd way behind a point and before all others of which however none would stand closest to the cut. One of the two lines into which the line would be split upon division would therefore have an end point, but the other no beginning point. This inference has been quite correctly drawn by Bolzano, who was led thereby to his monstrous doctrine that there would exist bodies with and without surfaces, the one class containing just so many as the other, because contact would be possible only between a body with a surface and another without. He ought, rather, to have had his attention drawn by such consequences to the fact that the whole conception of the line and of other continua as sets of points runs counter to the concept of contact and thereby abolishes precisely what makes up the essence of the continuum. (1988, p. 105)

boundaries that have only partial pleroses up to a point in which the sum of their pleroses adds up to 1 or full plerosis. On a purely extensional account, like the set-theoretical one, two boundaries which occupy the same region of space are to be identified as a single entity. This is the reason why one has to make the arbitrary decision as to whether, given a 3-dimensional region, its boundary is to be thought of as a part of the region or as a part of its complement. On the other hand, with this possibility of a coincidence of boundaries, then, one can *define* the notion of touching as being the relation holding between two regions of space that have at least one pair of boundaries with partial pleroses which coincide at least partially. So, on this account, both the original region and its complement would have their boundaries as parts. But each boundary would have half plerosis in opposite directions and, thus, could coincide.

This new definition bypasses much of the intrinsic unintuitiveness of the set-theoretic account. As we mentioned, in this picture, we do not have to make an arbitrary decision as to whether "the common boundary" is part of the first or the second touching regions; much on the contrary, in it we don't have open regions at all (even partially), for every proper region[27] has a boundary, which in general will have half plerosis (in the case of 2-dimensional boundaries of a 3-dimensional region in 3-dimensional space) and, therefore, will be able to coincide with other boundaries having partial pleroses, thus creating the alleged contact between these two regions.

There is more to Brentano's characterization of continua, but this is enough for our purposes here. It is, as we mentioned in the beginning, essentially an account of something like "the continua that are presented to us in spatio-temporal intuition" and *not* an abstract and fully general

[27] I.e., any region that is a proper subregion of the whole embedding space.

account of what it is to be a continuum. Thus, with the help of the later developments in mathematical topology, we can characterize his account, not as a brand new conception of topology as a general discipline, but certainly as a very interesting and self-consistent account of something like the "real" or "phenomenological" topology of the "real" or "phenomenologically present" continua in spatial-temporal intuition. And one must certainly say that, in this restricted scope, it surely succeeds as a very interesting and consistent account, having many interesting connection points with issues in ontology, such as *e.g.* issues regarding the notion of contact.

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Identifying the birth of contemporary philosophy involves critically reviewing ideas that were widely disseminated at the beginning of analytic philosophy and phenomenology. In both cases, rigorous historiography has been replaced by "creation myths" that, although relevant to determining the identity that each of the currents attributes to itself, ignore a web of complex systematic and historical relationships. This is the scope of the book Origins of Contemporary Philosophy: Studies on the Philosophy of Mathematics, published here by "Apolodoro Virtual Edições" as a special issue of the series "Rationality, Intentionality and Semantics," which brings together a presentation of the research group "Origins of Contemporary Philosophy" (GPOFC/PUC-SP) by Prof. Dr. Evandro O. Brito (UNICENTRO) and three papers resulting from research carried out by the German researcher Prof. Dr. Julia Franke-Reddig (University of Geneva/University of Siegen) and the Brazilian researchers Prof. M.e. Ernesto Maria Giusti (UNICENTRO) and Prof. Dr. Arthur Heller Britto (Pontifical Catholic University of São Paulo).

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